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# Thermodynamic Limit of the $q$-State Potts–Hopfield Model with Infinitely Many Patterns

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We prove the almost sure convergence of the free energy and of the overlap order parameters in a $q$-state version of the Hopfield neural network model. We compute explicitly these limits for all temperatures different from some critical value. The number of stored patterns is allowed to grow with the size of the system $N$ like $(\alpha/\ln q) \ln N$. We study the limiting behavior of the extremal states of the model that are the measures induced on the Gibbs measures by the overlap parameters.

**KEY WORDS:** Neural networks; disordered systems; mean field theory; Potts models.

## 1. INTRODUCTION

In this paper we study the thermodynamic of the $q$-state Potts–Hopfield model for neural networks. This model generalizes the standard Hopfield model, which serves to store binary ("black and white") patterns of information, to one in which "colored" patterns, that is, patterns for which each site can be in $q$ different states, are stored. Such models have been proposed in refs. 3, 8, 10, and 11.

We consider a network made of $N$ neurons (sites) which we label by the set $\Lambda = \{1,..., N\}$. Each neuron is allowed to be in $q$ different states and we denote by $\mathscr{S} = \{1,..., q\}$ the state space of a single neuron. A state of the neural network is thus described by $\sigma = \{\sigma_i\}_{i \in \Lambda}$, where $\sigma_i$ takes value in $\mathscr{S}$. We may also think as $\sigma$ as a spin configuration on $\Lambda$.

A given set of $p$ spin configurations, denoted by $\xi^1,..., \xi^p \in \mathscr{S}^\Lambda$, will be

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chosen as the original patterns we want to memorize. To this purpose, we define the energy function

$$H_\Lambda(\sigma;\xi) = -\frac{1}{2N}\sum_{\mu=1}^{p}\left\{\sum_{i=1}^{N}\frac{q}{q-1}\left[\delta(\xi_i^\mu,\sigma_i)-\frac{1}{q}\right]\right\}^2 \tag{1.1}$$

where $\delta(\cdot,\cdot)$ is the Kronecker symbol and $\xi$ denotes the family $\xi^1,\dots,\xi^p$.

Obviously we cannot study this model for any possible choice of patterns and the best we may hope for is to obtain statements that are valid for all "typical" choices of patterns. To make this notion more precise, one is led to introduce a probability space $(\Omega, \Sigma, \mathbb{P})$ such that there exists a family $(\xi_i^\mu)_{i\in\mathbb{N},\mu\in\mathbb{N}}$ of independent, identically distributed (i.i.d.) random variables with $\mathbb{P}(\xi_i^\mu=\gamma)=1/q$ for all $\gamma\in\{1,\dots,q\}$, $i\in\mathbb{N}$ and $\mu\in\mathbb{N}$. We will denote by $\xi$ the previous family and by $\xi|_{N,p}$ the restriction $\xi|_{N,p}=(\xi_i^\mu)_{i=1,\dots,N;\mu=1,\dots,p}$. If there is no danger of confusion we will also denote by $\xi$ this restriction. We will thus consider each $H_\Lambda(\sigma;\xi)$ as a random variable on the probability space $(\Omega,\Sigma,\mathbb{P})$. The idea is that for fixed $\Lambda$, $H_\Lambda(\sigma;\xi)$ as a function of $\sigma$ should take its minima when $\sigma$ is one of the prestored patterns $\xi^\mu$. It has been proven in fact in ref. 15 for $q=2$ and in ref. 8 for $q\geqslant 2$ that there exist at least one local minimum about each pattern $\xi^\mu$. That is to say, for $p<\alpha N$ and $\alpha\equiv\alpha(q)$ sufficiently small, this minimum is taken on at a configuration $\sigma$ for which the discrepancy with respect to $\xi^\mu$ is a small fraction of $N$ depending on $\alpha$.

The retrieval process consists in introducing a Markov process (dynamic) on the state space $\mathcal{S}^\Lambda$ in such a way that the associated invariant measure is given by the Gibbs distribution $\{\exp[-\beta H_\Lambda(\sigma;\xi)]\}/Z_\Lambda(\xi)$ corresponding to the Hamiltonian (1.1). The inverse temperature $\beta$ parametrizes the noise of the memory. Since we are interested in asymptotic properties of very large networks, we will investigate the thermodynamic limit of these distributions.

The thermodynamic formalism for the standard Hopfield model ($q=2$) has been introduced in ref. 2 (see also refs. 1, 10, and 14) and developed by Koch and Piasko[12] in the particular form that we will follow here. In ref. 12 they obtained an exact expression for the free energy and the overlap of the local magnetization with the stored patterns in the limit $N\uparrow\infty$ for all temperatures, provided the number of stored patterns is bounded by $p(N)\leqslant(\alpha/\ln 2)\ln N$. The essential idea behind their method is a clever change of variables (originally introduced by Grensing and Kühn[9]) that allows one to write an effective Hamiltonian that is a sum of Curie–Weiss Hamiltonians. On a certain subset of the probability space of the patterns (we will show here that this subset actually has measure one) one can then evaluate exactly the thermodynamic quantities of this model.

Notice that here the patterns correspond to exact global minimas of the free energy functional. Moreover, the patterns retrieved under the retrievial dynamic are expected to be the exact prestored patterns, in contrast with refs. 15 and 8, where errors are allowed.

Generalizing this approach to the $q$-state model requires us to deal with some hew geometric complexities and in particular needs the solution of the Curie–Weiss version of the Potts model. This latter has been given in a recent paper by Kesten and Schonmann.[13] We give a self-contained derivation of it for our purposes in Section 3.

The remainder of this paper is organized as follows. In Section 2 we describe the change of variables leading to an effective Hamiltonian amenable to a mean field treatment. Section 3 derives the exact solution of the mean field Potts model that will provide the basis of the subsequent analysis. In Section 4 we define the free energy for our model and prove the first central result of this paper (Theorem 1): If the number of stored patterns is given as some function $p(N)$ of $N$ that satisfies the bound $p(N) \leqslant (\alpha/\ln q) \ln N$ with $\alpha < 1$, then the limit as $N \uparrow \infty$ of the free energy exists for almost all (with respect to the probability measure on the space of patterns introduced above) choices of patterns $\xi$. Moreover, the limiting function $\tilde{F}(\beta)$ is independent of $\xi$ and will be computed explicitly. In Section 5 we investigate in more detail the structure of the infinite-volume Gibbs states. We will add a magnetic field term $\eta \sum_i [\delta(\xi_i^\alpha, \sigma_i) - 1/q]$ to the Hamiltonian that "favors" one particular pattern $\xi^\alpha$. We then introduce the order parameters

$$m_\Lambda^\mu(\sigma; \xi) = \frac{1}{N} \sum_{i=1}^{N} \left[ \delta(\xi_i^\mu, \sigma_i) - \frac{1}{q} \right], \qquad \mu = 1,..., p$$

which is the overlap between a configuration $\sigma$ and the pattern $\xi^\mu$, and

$$m_\Lambda^{\mu,\alpha}(\xi) = \langle m_\Lambda^\mu(\sigma; \xi) \rangle_\alpha, \qquad \mu = 1,..., p$$

which is their expectation with respect to the Gibbs measure $\mathscr{G}^\alpha$ in the presence of the field coupling to the pattern $\xi^\alpha$. We will show (Theorem 2) that again for almost all choices of patterns the limits of these quantities exist. Moreover, as $\eta \downarrow 0$ and for $\beta$ different from some critical value $\beta^c$,

$$m^{\mu,\alpha} = \frac{q-1}{q} s_0(\beta)\, \delta(\alpha, \mu) \geqslant 0$$

where $s_0(\beta)$ has a jump at the critical temperature (thus the model exhibits a first-order phase transition in the mean overlap parameter). As a consequence of Theorem 2, we will show (Theorem 3) that the measure $\mathscr{Q}^\alpha$

induced by the Gibbs measures $\mathscr{G}^{\alpha}$ on the overlaps converge to a Dirac distribution concentrated on $m^{\mu,\alpha}$. These induced measures, first studied in the Curie–Weiss model in ref. 7 for the magnetization parameter, are the extremal states of our model. That is (Theorem 4), adding a magnetic field coupling to a finite number $l$ of patterns with equal strength, the measure $\mathscr{Q}^{\alpha_1,\dots,\alpha_l}$ induced by the Gibbs measure $\mathscr{G}^{\alpha_1,\dots,\alpha_l}$ on the overlaps converges to a symmetric linear combination of the extremal states $\mathscr{Q}^{\alpha_r}$, $r=1,\dots,l$.

The results obtained here present a fairly complete analysis of the thermodynamic limit of the Potts–Hopfield model in the case of a "low density of patterns." The dynamical properties of our model are, however, also affected by local (and not global) minima of the Curie–Weiss free energy functionals. Their structure will be investigated in a followup article.

## 2. AN EFFECTIVE HAMILTONIAN

For a given choice of patterns $\xi$, the Hamiltonian (1.1) will take the same value for many spin configurations. To make this fact evident, we will perform a change of variables. This will allow us to introduce an effective Hamiltonian depending on a reduce number of degrees of freedom and that will be convenient for a mean field treatment. This transformation has been introduced in ref. 9 and used in the standard Hopfield model in ref. 12.

Using the formula

$$\delta(\xi_i^\mu, \sigma_i) = \sum_{\gamma=1}^{q} \delta(\xi_i^\mu, \gamma)\, \delta(\gamma, \sigma_i) \tag{2.1}$$

we have

$$\delta(\xi_i^\mu, \sigma_i) - \frac{1}{q} = \sum_{\gamma=1}^{q} \left[\delta(\xi_i^\mu, \gamma) - \frac{1}{q}\right]\left[\delta(\gamma, \sigma_i) - \frac{1}{q}\right] \tag{2.2}$$

and the Hamiltonian (1.1) can be rewritten as

$$H_\Lambda(\sigma; \xi) = -\frac{1}{2N} \sum_{\mu=1}^{p} \left\{\sum_{i=1}^{N} \sum_{\gamma=1}^{q} \frac{q}{q-1} \left[\delta(\xi_i^\mu, \gamma) - \frac{1}{q}\right]\left[\delta(\gamma, \sigma_i) - \frac{1}{q}\right]\right\}^2 \tag{2.3}$$

Let us consider $\xi|_{N,p}$ as a $p \times N$ matrix $\{\xi_i^\mu\}_{\mu=1,\dots,p; i\in\Lambda}$, that is, a map from $\mathbb{Z}_p \times \mathbb{Z}_N$ into $\mathscr{S}$. The row and column vectors $\xi^\mu$ and $\xi_i$ can then be viewed as given by the maps $\underline{\xi}$ and $\hat{\xi}$ from $\mathbb{Z}_p \times \mathbb{Z}_N$ into $(\mathscr{S}^N, \mathbb{Z}_N)$ and $(\mathscr{S}^p, \mathrm{Z}_p)$ respectively. That is,

$$\underline{\xi}:\quad (\mu, i) \to (\{\xi_j^\mu\}_{j\in\Lambda}, i)$$

and

$$\hat{\xi}:\quad (\mu, i) \to (\{\xi_i^\nu\}_{\nu=1,\dots,p}, \mu)$$

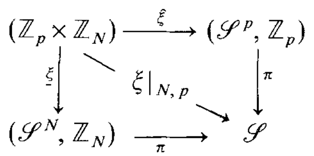
Let $\{e_1, \ldots, e_d\}$ be an arbitrarily fixed enumeration of the $d = q^p$ vectors of $\mathscr{S}^p$. As before we can consider the $p \times d$ matrix $\{e_k^\mu\}_{\mu=1,\ldots,p;\, k=1,\ldots,d}$ as a map from $\mathbb{Z}_p \times \mathbb{Z}_d$ into $\mathscr{S}$ and the maps $\underset{\sim}{e}$ and $\hat{e}$ from $\mathbb{Z}_p \times \mathbb{Z}_d$ into $(\mathscr{S}^d, \mathbb{Z}_d)$ and $(\mathscr{S}^p, \mathbb{Z}_p)$ as giving the row and column vectors $e^\mu$ and $e_k$:

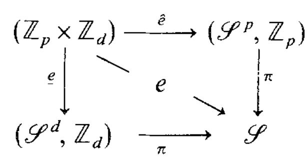
$$\underset{\sim}{e}\colon \quad (\mu, k) \to (\{e_j^\mu\}_{j=1,\ldots,d}, k)$$

and

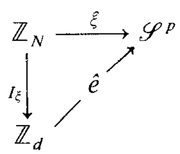
$$\hat{e}\colon \quad (\mu, k) \to (\{e_k^\nu\}_{\nu=1,\ldots,p}, \mu)$$

These mappings are illustrated by the following diagrams:





where $\pi$ is the canonical projection $\pi(\xi_i, \mu) = \xi_i^\mu$, etc. Note that, e.g., the action of $\hat{\xi}$ on the $\mathbb{Z}_p$ functions is trivial; we will thus think of $\hat{\xi}$, $\underset{\sim}{\xi}$, etc., also as maps from $\mathbb{Z}_N$ into $\mathscr{S}^p$, and $\mathbb{Z}_N$ into $\mathscr{S}^N$. Notice also that the map $\hat{e}$ is invertible. As a consequence, for a fixed $e$ and any given $\xi|_{N,p}$, the two mappings $\hat{\xi}$ and $\hat{e}$ define a map $I_\xi \equiv \hat{e}^{-1} \circ \hat{\xi}$ from $\mathbb{Z}_N$ into $\mathbb{Z}_d$ described by the following diagram:



Now, notice that $I_\xi$ induces a partition of the set $\Lambda$ into $d$ disjoint (possibly empty) subsets $\Lambda_k(\xi) = \{i \in \Lambda\colon I_\xi(i) = k\}$ with the property that, for all $i \in \Lambda_k(\xi)$, $\xi_i = e_k$. By using this partition, the Hamiltonian (2.3) becomes

$$\begin{aligned} H_\Lambda(\sigma; \xi) &= -\frac{1}{2N} \sum_{\mu=1}^{p} \left\{ \sum_{k=1}^{d} \sum_{i \in \Lambda_k(\xi)} \sum_{\gamma=1}^{q} \frac{q}{q-1} \left[ \delta(\xi_i^\mu, \gamma) - \frac{1}{q} \right] \left[ \delta(\gamma, \sigma_i) - \frac{1}{q} \right] \right\}^2 \\ &= -\frac{1}{2N} \sum_{\mu=1}^{p} \left\{ \sum_{k=1}^{d} \sum_{\gamma=1}^{q} \frac{q}{q-1} \left[ \delta(e_k^\mu, \gamma) - \frac{1}{q} \right] \sum_{i \in \Lambda_k(\xi)} \left[ \delta(\gamma, \sigma_i) - \frac{1}{q} \right] \right\}^2 \end{aligned} \tag{2.4}$$

This last expression makes it evident that the Hamiltonian only depends on the variables

$$y_k^{(\gamma)} \equiv \sum_{i \in \Lambda_k(\xi)} \left[ \delta(\gamma, \sigma_i) - \frac{1}{q} \right]$$

Let $L$ denote the vector $(L_1,..., L_d)$ in $\mathbb{R}^d$ whose components $L_k = |\Lambda_k(\xi)|$ are given by the cardinality of the subsets $\Lambda_k(\xi)$. Then, for all $\gamma = 1,..., q$, $y_k^{(\gamma)}$ takes values in $\{-L_k/q, 1 - L_k/q,..., L_k - L_k/q\}$. Also notice that, considering $y^{(\gamma)}$ as vectors in $\mathbb{R}^d$, they satisfy by construction the constraint

$$\sum_{\gamma=1}^{q} y^{(\gamma)} = 0 \tag{2.5}$$

We denote by $Y_\xi$ the set $\{y^{(\gamma)}\}_{\gamma=1,...,q}$ of these $q$ vectors, and the space of all possible $Y_\xi$ will be called $\Xi_\xi$. Through the $Y_\xi$, the map $I_\xi$ also induces a partition of the state space $\mathscr{S}^\Lambda$ into subspaces

$$\mathscr{S}(Y_\xi) = \left\{\sigma \in \mathscr{S}^\Lambda \,\middle|\, \sum_{i \in \Lambda_k(\xi)} \left[\delta(\gamma, \sigma_i) - \frac{1}{q}\right] = y_k^{(\gamma)};\ 1 \leqslant \gamma \leqslant q,\ 1 \leqslant k \leqslant d\right\}$$

If now we still define the vectors $\varepsilon^{\mu,(\gamma)}$ in $\mathbb{R}^d$ by $\varepsilon_k^{\mu,(\gamma)} = \delta(e_k^\mu, \gamma) - 1/q$, our Hamiltonian finally takes the desired reduced form: for all $\sigma \in \mathscr{S}(Y_\xi)$

$$H_\Lambda(\sigma; \xi) = -\frac{1}{2N} \sum_{\mu=1}^{p} \left[\sum_{\gamma=1}^{q} \frac{q}{q-1} \langle \varepsilon^{\mu,(\gamma)} | y^{(\gamma)} \rangle_d\right]^2 \tag{2.6}$$

Here, $\langle \cdot | \cdot \rangle_d$ denotes the scalar product in $\mathbb{R}^d$ and $\| \cdot \|_d$ will denote the corresponding norm.

Our next goal is to obtain an effective Hamiltonian $\bar{H}_\Lambda(Y_\xi)$ for the variables $Y_\xi$. To do this, we write the partition function

$$\begin{aligned}
Z_{\Lambda,\beta}(\xi) &= \frac{1}{q^N} \sum_{\sigma \in \mathscr{S}^\Lambda} \exp[-\beta H_\Lambda(\sigma; \xi)] \\
&= \frac{1}{q^N} \sum_{Y_\xi \in \Xi_\xi} \sum_{\sigma \in \mathscr{S}(Y_\xi)} \exp[-\beta H_\Lambda(\sigma; \xi)] \\
&= \frac{1}{q^N} \sum_{Y_\xi \in \Xi_\xi} \left\{\exp\left[\frac{\beta}{2N} \sum_{\mu=1}^{p} \left(\sum_{\gamma=1}^{q} \frac{q}{q-1} \langle \varepsilon^{\mu,(\gamma)} | y^{(\gamma)} \rangle_d\right)^2\right]\right\} \sum_{\sigma \in \mathscr{S}(Y_\xi)} 1 \\
&= \frac{1}{q^N} \sum_{Y_\xi \in \Xi_\xi} \exp\left[\frac{\beta}{2N} \sum_{\mu=1}^{p} \left(\sum_{\gamma=1}^{q} \frac{q}{q-1} \langle \varepsilon^{\mu,(\gamma)} | y^{(\gamma)} \rangle_d\right)^2 + \ln |\mathscr{S}(Y_\xi)|\right] \\
&\equiv \frac{1}{q^N} \sum_{Y_\xi \in \Xi_\xi} \exp[-\beta \bar{H}_\Lambda(Y_\xi)]
\end{aligned} \tag{2.7}$$

where $\bar{H}(Y_\xi)$ is defined by

$$\bar{H}(Y_\xi) = -\frac{1}{2N} \sum_{\mu=1}^{p} \left[\sum_{\gamma=1}^{q} \frac{q}{q-1} \langle \varepsilon^{\mu,(\gamma)} | y^{(\gamma)} \rangle_d\right]^2 - \frac{1}{\beta} \ln |\mathscr{S}(Y_\xi)| \tag{2.8}$$

and $|\mathscr{S}(Y_\xi)|$ is given by

$$|\mathscr{S}(Y_\xi)| = \prod_{k=1}^{d} \left[ \frac{L_k!}{\prod_{\gamma=1}^{q} (y_k^{(\gamma)} + L_k/q)!} \right] \tag{2.9}$$

The following lemma collects some properties on the vectors $\varepsilon^{\mu,(\gamma)}$ appearing in $\bar{H}(Y_\xi)$.

**Lemma 2.1.** Let $\mu, \nu \in \{1,..., p\}$, $\alpha, \gamma \in \{1,..., q\}$, and $\mathbb{1}_d \equiv (1,..., 1) \in \mathbb{R}^d$. Then:

(i) $\langle \mathbb{1}_d | \varepsilon^{\mu,(\gamma)} \rangle_d = 0$.

(ii) $\sum_{\gamma=1}^{q} \varepsilon^{\mu,(\gamma)} = 0$.

(iii) If $\mu \neq \nu$, then $\langle \varepsilon^{\mu,(\gamma)} | \varepsilon^{\nu,(\alpha)} \rangle_d = 0$ for all $\alpha, \gamma$.

(iv) If $\mu = \nu$, then

$$\langle \varepsilon^{\mu,(\gamma)} | \varepsilon^{\mu,(\alpha)} \rangle_d = \begin{cases} c_1 & \text{if} \quad \alpha = \gamma \\ c_2 & \text{if} \quad \alpha \neq \gamma \end{cases}$$

where $c_1 = d(q-1)/q^2$ and $c_2 = -d/q^2$.

*Proof.* These properties are easily verified by using the definitions of the $\varepsilon^{\mu,(\gamma)}$ and remembering that $\{e_1,..., e_d\}$ is a fixed enumeration of the $d = q^p$ vectors of $\mathscr{S}^p$. For example, (i) simply follows from the equality $\sum_{k=1}^{d} \delta(e_k^\mu, \gamma) = q^{p-1} = d/q$ ∎.

We now introduce the projection operators

$$P^{(\gamma)} \equiv \frac{1}{c_1} \sum_{\mu=1}^{p} |\varepsilon^{\mu,(\gamma)}\rangle_d \langle \varepsilon^{\mu,(\gamma)}|_d, \qquad \gamma = 1,..., q$$

which project orthogonally onto the subspaces $V^{(\gamma)}$ spanned by the vectors $\{\varepsilon^{\mu,(\gamma)}\}_{\mu=1,...,p}$. Norice that, however, these subspaces are not orthogonal to each other.

After some lengthy but easy calculations and making use of Lemma 2.1, we can express the Hamiltonian $\bar{H}_\Lambda(Y_\xi)$ in terms of this projection as

$$\begin{aligned} \bar{H}_\Lambda(Y_\xi) = & \frac{d}{2N} \left[ \left\| \sum_{\gamma=1}^{q} P^{(\gamma)} y^{(\gamma)} \right\|_d^2 - \frac{q}{q-1} \sum_{\gamma=1}^{q} \| P^{(\gamma)} y^{(\gamma)} \|_d^2 \right] \\ & - \frac{1}{\beta} \sum_{k=1}^{d} \ln \left[ \frac{L_k!}{\prod_{\gamma=1}^{q} (y_k^{(\gamma)} + L_k/q)!} \right] \end{aligned} \tag{2.10}$$

where we have used (2.9).

In terms of the Hamiltonian $\bar{H}_\Lambda(Y_\xi)$ we may write the finite-volume free energy as

$$F_\Lambda(\xi) = -\frac{1}{N} \ln \sum_{Y_\xi \in \Xi_\xi} \frac{\exp[-\beta \bar{H}_\Lambda(Y_\xi)]}{q^N} \tag{2.11}$$

We will study the thermodynamic limit of this quantity in Section 4.

## 3. MEAN-FIELD THEORY FOR POTTS MODEL

A major ingredient for the analysis of the effective Hamiltonian introduced in the previous section will be the mean-field theory for Potts models that has been developed recently by Kesten and Schonmann.[13] In this section we will summarize the results that we will need later.

Using the same notations as in Section 2, we define the Curie–Weiss Hamiltonian for the Potts model by:

$$H_{\Lambda,h}^{(\mathrm{C.W.})}(\sigma) = -\frac{\mathscr{J}}{2\,|\Lambda|} \sum_{i,j \in \Lambda} \delta(\sigma_i, \sigma_j) - \sum_{\gamma=1}^{q} h_\gamma \sum_{i \in \Lambda} \delta(\sigma_i, \gamma) \tag{3.1}$$

where the interaction couples each pair of spins in $\Lambda$ with equal strength $\mathscr{J} > 0$, and $h = (h_1,..., h_q)$, $h_\gamma \geqslant 0$ for all $\gamma$, is an external magnetic field. Introducing the vectors $x_i = (\delta(\sigma_i, 1),..., \delta(\sigma_i, q))$ aligned on the axes of $\mathbb{R}^q$ and using formula (2.1), we have

$$\sum_{i,j \in \Lambda} \delta(\sigma_i, \sigma_j) = \left\| \sum_{i=1}^{N} x_i \right\|_q^2, \qquad \sum_{\gamma=1}^{q} h_\gamma \sum_{i \in \Lambda} \delta(\sigma_i, \gamma) = \left\langle h \,\middle|\, \sum_{i=1}^{N} x_i \right\rangle_q$$

which allow us to write the Hamiltonian (3.1) as a function of the sum $z \equiv (1/N) \sum_{i=1}^{N} x_i$,

$$H_{\Lambda,h}^{(\mathrm{C.W.})}(\sigma) = -N \left[ \frac{\mathscr{J}}{2} \|z\|_q^2 + \langle h \mid z \rangle_q \right] \tag{3.2}$$

where the vectors $z = (z_1,..., z_q)$ are in the subset $\Delta_q$ of $\mathbb{R}^q$ given by

$$\Delta_q = \left\{ z \in \mathbb{R}^q \,\middle|\, z_\gamma \geqslant 0,\ \sum_{\gamma=1}^{q} z_\gamma = 1 \right\} \tag{3.3}$$

The partition function associated to this model is given by

$$Z_\Lambda^{(\mathrm{C.W.})}(\beta, h) = \int_{\mathscr{S}^\Lambda} \mu_\Lambda(d\sigma) \exp[-\beta H_{\Lambda,h}^{(\mathrm{C.W.})}(\sigma)] \tag{3.4}$$

where $\mu_\Lambda(d\sigma) = \prod_{i\in\Lambda} \rho(d\sigma_i)$ and $\rho$ is the uniform distribution on $\mathscr{S}$. Then defining the infinite-volume free energy as

$$F^{(\mathrm{C.W.})}(\beta, h) = \lim_{N\to\infty} -\frac{1}{N} \ln Z_\Lambda^{(\mathrm{C.W.})}(\beta, h) \tag{3.5}$$

we have the following result.

**Lemma 3.1.:**

$$F^{(\mathrm{C.W.})}(\beta, h) = \min_{z\in\Delta_q} \sum_{\gamma=1}^{q} \left\{ -\frac{\beta}{2} z_\gamma^2 + z_\gamma \ln q z_\gamma - h_\gamma z_\gamma \right\} \tag{3.6}$$

*Remark.* The proof of Lemma 3.1 is essentially contained in ref. 13. To be self-contained, we give here an alternative proof.

*Proof.* Using (3.2), the partition function can be rewritten as

$$Z_\Lambda^{(\mathrm{C.W.})}(\beta, h) = \int_{\Delta_q} \tilde{\mu}_\Lambda(dz) \exp\left\{\beta N \left[\frac{\mathscr{I}}{2} \|z\|_q^2 + \langle h \mid z\rangle_q\right]\right\}$$

where $\tilde{\mu}_\Lambda$ denotes the distribution of $z$, i.e., the distribution of the mean $(1/N)\sum_{i=1}^{N} x_i$ of $N$ independent random vectors $x_i$ with common distribution $\mathbb{P}(x_i = e_\gamma) = 1/q$, $\gamma = 1,..., q$, where $e_1,..., e_q$ are the axes of $\mathbb{R}^q$. It follows by Cramer's theorem[4,16] that the distributions $\{\tilde{\mu}_\Lambda, N = 1, 2,...\}$ satisfy the large-deviation property with rate function

$$I(z) = \max_{t\in\Delta_q} \{\langle t \mid z\rangle_q - \ln M(t)\}$$

where $M(t)$ is the moment-generating function

$$M(t) = \mathbb{E}(e^{\langle t \mid x_i\rangle_q}) = \sum_{\gamma=1}^{q} \frac{e^{t_\gamma}}{q}$$

A simple calculation yields

$$I(z) = \begin{cases} \infty & \text{if } z \notin \Delta_q \\ \sum_{\gamma=1}^{q} z_\gamma \ln(q z_\gamma) & \text{if } z \in \Delta_q \end{cases}$$

Now, since the function $\frac{1}{2}\mathscr{I} \|z\|_q^2 + \langle h \mid z\rangle_q$ is bounded and continuous on $\Delta^q$ we have by the large-deviation principle[6,16]

$$\lim_{N\to\infty} -\frac{1}{N}\ln\int_{\Delta_q} \tilde{\mu}_\Lambda(dz)\exp\left\{\beta N\left[\frac{\mathscr{J}}{2}\|z\|_q^2+\langle h|z\rangle_q\right]\right\}$$
$$=\min_{z\in\Delta_q}\left\{-\frac{\mathscr{J}}{2}\|z\|_q^2-\langle h|z\rangle_q+I(z)\right\}$$

which proves the lemma. ∎

The problem now is to study the right-hand side of (3.6). Kesten and Schonmann proved the following result.

**Lemma 3.2.** Let $h=0$. Then the minimum in the right-hand side of (3.6) is taken on only at a $z$ of the form

$$z_1=\frac{1}{q}[1+(q-1)s], \qquad z_\gamma=\frac{1}{q}(1-s), \qquad 2\leqslant\gamma\leqslant q \tag{3.7}$$

for some $0\leqslant s\leqslant 1$ or at a point obtained from such a $z$ by permuting the coordinates.

Let $\beta^c$ denote the critical value of $\beta$:

$$\beta^c=\begin{cases}2(q-1/q-2)\ln q & \text{if } q>2\\ 2 & \text{if } q=2\end{cases} \tag{3.8}$$

If $\beta>\beta^c$, then the minimum in (3.6) is taken on for $s=s_0(\beta)$, where $s_0(\beta)$ is the largest solution of

$$s_0=\frac{1-e^{-\beta s_0}}{1+(q-1)e^{-\beta s_0}} \tag{3.9}$$

If $\beta<\beta^c$, then the unique $z$ at which the minimum is taken on in (3.6) is

$$z_\gamma=\frac{1}{q}, \qquad 1\leqslant\gamma\leqslant q \tag{3.10}$$

corresponding to $s=s_0=0$ in (3.7). Finally,

$$s_0(\beta)>0 \quad \text{for} \quad \beta>\beta^c \qquad \text{and} \qquad \lim_{\beta\downarrow\beta^c} s_0(\beta)=\frac{q-2}{q-1}$$

and

$$s_0(\beta) \quad \text{is differentiable in} \quad \beta \quad \text{for} \quad \beta>\beta^c$$

Notice that in the case $q > 2$, $s_0(\beta)$ has a jump discontinuity at $\beta = \beta^c$, which mean that the model has a first-order phase transition, in contrast to the case $q = 2$, where the transition is second order.

For later use we will denote the $q$ minima of the right-hand side of (3.6) by $\tilde{z}^{(\gamma)}$.

## 4. THERMODYNAMIC LIMIT OF THE FREE ENERGY

In this section we will prove the almost sure convergence of the free energy when the number $N$ of neurons tends to infinity and compute explicitly this limit. The number of stored patterns is allowed to grow to infinity; however, we require it to satisfy the bound $p < (\alpha/\ln q) \ln N$, where $0 \leqslant \alpha < 1$ can be chosen arbitrarily. From now on, $p \equiv p(N)$ will be assumed to be a given such function.

*Remark.* We show in the Appendix that the almost sure convergence of the free energy extends easily to the convergence in the spaces $L^r(\Omega, \Sigma, \mathbb{P})$, $1 \leqslant r < \infty$.

In order to give a precise formulation to the theorem, we recall that $\xi$ denotes the infinite family $\xi = (\xi_i^\mu)_{i \in \mathbb{N}, \mu \in \mathbb{N}}$ of i.i.d. random variables on the probability space $(\Omega, \Sigma, \mathbb{P})$ and that $\xi|_{N,p}$ denotes the restriction $\xi|_{N,p} = (\xi_i^\mu)_{i=1,\dots,N; \mu=1,\dots,p}$. Remembering that $\xi|_{N,p}$ induces a random partition of $\Lambda$ in $d$ boxes $\Lambda_k(\xi|_{N,p})$ of length $L_k(\xi|_{N,p})$, we can introduce the subsets

$$\mathscr{E}^{N,p} = \left\{ \omega \in \Omega \mid L_k = \frac{N}{d}(1 + \lambda_k), \; |\lambda_k| < \delta_N, \; 1 \leqslant k \leqslant d \right\}$$

where $\delta_N \equiv (d/N)^{1/2} \ln N$. Notice that $\delta_N$ is a decreasing function of $N$, which means that, for $N$ large enough, each $L_k$ is constrained to be near its mean value. The reason for this definition is that on these subsets the free energy of our model can be computed using the mean-field theory developed in Section 3.

On the other hand, almost all $\omega$ will eventually, for $N$ large enough (how large will depend on the sample), be contained in the subsets $\mathscr{E}^{N,p}$. To make this notion precise, we define the subset $\tilde{\Omega} \subset \Omega$ in the following way:

$$\tilde{\Omega} = \{ \omega \in \Omega \mid \exists N_0 : \forall N > N_0, \; \omega \in \mathscr{E}^{N,p} \}$$

We then have the following result.

**Proposition 4.1:**

$$\mathbb{P}(\tilde{\Omega}) = 1 \tag{4.1}$$

To prove this proposition, we need the following lemma.

**Lemma 4.2.** Let $A_N$ be the event $A_N = \{\omega \in \Omega \mid \omega \notin \mathscr{E}^{N,p}\}$. Then

$$\mathbb{P}(A_N) \leqslant 2d \exp\left[-\frac{N}{2d}\left(1-\frac{1}{d}\right)^{-1} \delta_N^2\right] \tag{4.2}$$

*Proof.* By definition we have

$$\mathbb{P}(A_N) = \mathbb{P}(\exists i \in \Lambda \colon |\lambda_i| > \delta_N)$$

Obviously

$$\mathbb{P}(\exists i \in \Lambda \colon |\lambda_i| > \delta) \leqslant d\mathbb{P}(|\lambda_1| > \delta)$$

Using the exponential Markov inequality,[5] we have

$$\begin{aligned} \mathbb{P}(|\lambda_1| > \delta) &\leqslant \mathbb{E}(e^{(\lambda_1 - \delta)t}) + \mathbb{E}(e^{-(\lambda_1 + \delta)t}) \\ &= e^{-\delta t}[\mathbb{E}(e^{\lambda_1 t}) + \mathbb{E}(e^{-\lambda_1 t})] \end{aligned} \tag{4.3}$$

Remembering that $L_k = (N/d)(1 + \lambda_k)$ and defining the multinomial coefficient as

$$\binom{N}{L_1, L_2,..., L_{d-1}} = \begin{cases} \dfrac{N!}{L_1!\, L_2! \cdots L_{d-1}!\, [N - (L_1 + L_2 + \cdots + L_{d-1})]!} \\ \quad \text{if } \; L_k \geqslant 0, \quad k = 1,..., d-1 \\ 0 \\ \quad \text{if } \; \sum\limits_{k=1}^{d-1} L_k > N \end{cases}$$

we find that the Laplace transform $\mathbb{E}(e^{\lambda_1 t})$ equals

$$\begin{aligned} \mathbb{E}(\exp(\lambda_1 t)) &= \sum_{L_1, L_2,..., L_{d-1}} \frac{1}{d^N} \binom{N}{L_1, L_2,..., L_{d-1}} \\ &\quad \times \exp\left[\frac{L_1 - N/d}{(N/d)^{1/2}} \left(\frac{d}{N}\right)^{1/2} t\right] \\ &= [\exp(-t)] \left(\frac{d-1+\exp[(d/N)\, t]}{d}\right)^N \end{aligned}$$

Therefore $\mathbb{E}(e^{\lambda_1 t})$ satisfies the bound

$$\mathbb{E}(\exp(\lambda_1 t)) \leqslant \exp\left[\frac{d}{2N}\left(1-\frac{1}{d}\right) t^2 + O\left(\left(\frac{d}{N}\right)^{3/2} t^3\right)\right]$$

and choosing $t$ in (4.3) as $t=(N/d)\,\delta(1-1/d)^{-1}$, we get

$$\mathbb{P}(|\lambda_1|>\delta)\leqslant 2\exp\left[-\frac{N}{2d}\left(1-\frac{1}{d}\right)^{-1}\delta^2+O\left(\left(\frac{N}{d}\right)^{3/2}\delta^3\right)\right]$$

which proves the lemma. ∎

*Proof of Proposition* 4.1. We have

$$\mathbb{P}(\tilde{\Omega})=1-\mathbb{P}(\tilde{\Omega}^c)$$

where

$$\tilde{\Omega}^c=\{\omega\in\Omega\,|\,\forall N_0<\infty,\exists N>N_0:\omega\notin\mathscr{E}^{N,p}\}$$

Thus we have

$$0\leqslant\mathbb{P}(\tilde{\Omega}^c)\leqslant\mathbb{P}(\overline{\lim_{N\to\infty}}\,A_N)$$

By the Borel–Cantelli lemma,[5] $\mathbb{P}(\overline{\lim}_{N\to\infty}A_N)=0$ if $\sum_N\mathbb{P}(A_N)<\infty$. Now, by Lemma 4.2, this last condition holds. Thus, the proposition is proven. ∎

Having shown that $\tilde{\Omega}$ is a subset of measure one, we will now compute the thermodynamic limit of $F_\Lambda$ on this subset. For $\omega$ in $\Omega$ we will understand that $F_\Lambda(\xi)\equiv F_\Lambda(\xi|_{N,p})$. Let $\chi_{[\omega\in\tilde{\Omega}]}$ denote the characteristic function of $\tilde{\Omega}$.

**Proposition 4.3.** Let $\tilde{\beta}=[q/(q-1)]\,\beta$. Then for all $\tilde{\beta}\neq\beta^c$

$$\lim_{N\to\infty}\chi_{[\omega\in\tilde{\Omega}]}F_{\Lambda,\beta}(\xi)=\chi_{[\omega\in\tilde{\Omega}]}\tilde{F}(\beta)\tag{4.4}$$

where

$$\begin{aligned}\tilde{F}(\beta)&=\min_{z\in\Delta_q}\sum_{\gamma=1}^{q}\left\{-\frac{\tilde{\beta}}{2}z_\gamma^2+z_\gamma\ln(qz_\gamma)\right\}+\frac{\tilde{\beta}}{2q}\\ \Delta_q&=\left\{z\in\mathbb{R}^q\,|\,z_\gamma\geqslant 0,\ \sum_{\gamma=1}^{q}z_\gamma=1\right\}\end{aligned}\tag{4.5}$$

and $\beta^c$ is the inverse critical temperature of the Curie–Weiss model for Potts variables.

In order to prove this proposition, we introduce the following change of variables:

$$\begin{aligned}L_k&=\frac{N}{d}(1+\lambda_k)\\ y_k^{(\gamma)}&=L_kx_k^{(\gamma)}\end{aligned}\tag{4.6}$$

and we will denote by $\lambda$ the vector $(\lambda_1,..., \lambda_d)$ in $\mathbb{R}^d$, by $M$ the diagonal $d\times d$ matrix with entries $M_{kk}=\lambda_k$, and by $X_\xi$ the set of vectors $\{x^{(\gamma)}\}_{\gamma=1,...,q}$, where $x^{(\gamma)}=(x_1^{(\gamma)},..., x_d^{(\gamma)})$. Also notice that $\lambda$ and $X_\xi$ satisfy the constraints

$$\begin{aligned} &\sum_{k=1}^{d} \lambda_k=0 \\ &\sum_{\gamma=1}^{q} x^{(\gamma)}=0 \end{aligned} \tag{4.7}$$

Let us now define the function $h_\Lambda(\beta, \lambda, X_\xi)$ as

$$\begin{aligned} h_\Lambda(\beta, \lambda, X_\xi)=\frac{\beta}{2}\left[\left\| \sum_{\gamma=1}^{q} P^{(\gamma)}(\mathbb{1}+M)\, x^{(\gamma)}\right\|_d^2-\frac{q}{q-1}\sum_{\gamma=1}^{q} \|P^{(\gamma)}(\mathbb{1}+M)\, x^{(\gamma)}\|_d^2\right] \\ +\sum_{k=1}^{d} (1+\lambda_k)\sum_{\gamma=1}^{q} \left(x_k^{(\gamma)}+\frac{1}{q}\right)\ln\left(x_k^{(\gamma)}+\frac{1}{q}\right) \end{aligned} \tag{4.8}$$

Then we have the following result.

**Lemma 4.4.** For all $\omega$ in $\tilde{\Omega}$ there exists $N_0$ such that, for all $N>N_0$,

$$\left|\beta\bar{H}_\Lambda(Y_\xi)-\frac{N}{d}h_\Lambda(\beta, \lambda, X_\xi)|_{\lambda=0}\right|\leqslant c(\beta, q)\,\delta_N N \tag{4.9}$$

for some constant $c(\beta, q)$ independant of $N$ and $p$.

*Proof.* By the Stirling formula and performing the change of variables (4.6), (2.9) gives

$$\ln|\Omega(Y_\xi)|=-\frac{N}{d}\sum_{k=1}^{d} (1+\lambda_k)\sum_{\gamma=1}^{q} \left(x_k^{(\gamma)}+\frac{1}{q}\right)\ln\left(x_k^{(\gamma)}+\frac{1}{q}\right)+\sum_{k=1}^{d} g_k(X_\xi) \tag{4.10}$$

where

$$|g_k(X_\xi)|\leqslant\frac{q-1}{2}(1+\ln L_k) \tag{4.11}$$

$\beta\bar{H}_\Lambda(Y_\xi)$ may thus be written as

$$\beta\bar{H}_\Lambda(Y_\xi)=\frac{N}{d}h_\Lambda(\beta, \lambda, X_\xi)+\sum_{k=1}^{d} g_k(X_\xi) \tag{4.12}$$

which implies

$$\left|\beta \bar{H}_\Lambda(Y_\xi) - \frac{N}{d} h_\Lambda(\beta, \lambda, X_\xi)|_{\lambda=0}\right| \leqslant \frac{N}{d} |h_\Lambda(\beta, \lambda, X_\xi) - h_\Lambda(\beta, 0, X_\xi)| + \sum_{k=1}^{d} |g_k(X_\xi)| \tag{4.13}$$

Using simply that $\|a+b\|^2 - \|a\|^2 = \langle b \mid (2a+b)\rangle$, we find

$$\begin{aligned}\frac{N}{d} &|h_\Lambda(\beta, \lambda, X_\xi) - h_\Lambda(\beta, 0, X_\xi)| \\ &= \frac{\beta N}{2d} \left| \left\langle \sum_{\gamma=1}^{q} P^{(\gamma)} M x^{(\gamma)} \middle| \sum_{\gamma=1}^{q} P^{(\gamma)}(2+M)\, x^{(\gamma)} \right\rangle_d \right. \\ &\quad \left. - \frac{q}{q-1} \sum_{\gamma=1}^{q} \langle P^{(\gamma)} M x^{(\gamma)} \mid P^{(\gamma)}(2+M)\, x^{(\gamma)} \rangle_d \right|\end{aligned}$$

Note that under the hypothesis of the lemma we have $|\lambda_k| \leqslant \delta_N$ and that $|x_k^{(\gamma)}| \leqslant 1 - 1/q$. Thus, some simple manipulations involving the Cauchy–Schwarz inequality and the contraction properties of the projectors $P^{(\gamma)}$ show that the last expression can be bounded by

$$\frac{\beta}{2}\left(\frac{q^3}{q-1}\right) \delta_N (2+\delta_N)\, N$$

Putting this bound together with the bound (4.11), we finally get

$$\begin{aligned}\frac{N}{d} |h_\Lambda(\beta, \lambda, X_\xi) - h_\Lambda(\beta, 0, X_\xi)| &\leqslant \frac{\beta}{2}\left(\frac{q^3}{q-1}\right) \delta_N (2+\delta_N)\, N + 2d\left(\frac{q-1}{2}\right) \ln N \\ &\leqslant \left[\frac{3}{2}\left(\frac{q^3}{q-1}\right)\beta + q - 1\right] \delta_N N\end{aligned}$$

where we have used that $\delta_N = (d/N)^{1/2} \ln N$ and $d \ln N \leqslant \delta_N N$. Thus the lemma is proven. ▮

Using the definition of the free energy from (2.11), we may write

$$\begin{aligned}\chi_{[\omega \in \tilde{\Omega}]} F_\Lambda(\xi) &= -\frac{1}{N} \chi_{[\omega \in \tilde{\Omega}]} \ln \sum_{Y_\xi \in \Xi_\xi} \frac{\exp[-\beta \bar{H}_\Lambda(Y_\xi)]}{q^N} \\ &= -\frac{1}{N} \chi_{[\omega \in \tilde{\Omega}]} \ln \sum_{Y_\xi \in \Xi_\xi} \frac{1}{q^N} \exp\left[ -\left\{ \beta \bar{H}_\Lambda(Y_\xi) \right.\right. \\ &\quad \left.\left. - \frac{N}{d} h_\Lambda(\beta, 0, X_\xi) \right\} - \frac{N}{d} h_\Lambda(\beta, 0, X_\xi) \right]\end{aligned}$$

and by Lemma 4.4

$$\chi_{[\omega\in\tilde{\Omega}]}F_{\Lambda}(\xi)\leqslant\delta_N c'(\beta,q)-\frac{1}{N}\chi_{[\omega\in\tilde{\Omega}]}\ln\sum_{Y_\xi\in\Xi_\xi}\frac{\exp\left[-(N/d)\,h_\Lambda(\beta,0,X_\xi)\right]}{q^N}$$

and

$$\chi_{[\omega\in\tilde{\Omega}]}F_{\Lambda}(\xi)\geqslant -\delta_N c'(\beta,q)-\frac{1}{N}\chi_{[\omega\in\tilde{\Omega}]}\ln\sum_{Y_\xi\in\Xi_\xi}\frac{\exp\left[-(N/d)\,h_\Lambda(\beta,0,X_\xi)\right]}{q^N}\tag{4.14}$$

for some constant $c'(\beta, q)$. Notice that $\delta_N$ tends to 0 as $N$ tends to infinity. We are thus left to study the quantity

$$\frac{1}{N}\chi_{[\omega\in\tilde{\Omega}]}\ln\sum_{Y_\xi\in\Xi_\xi}\frac{\exp\left[-(N/d)\,h_\Lambda(\beta,0,X_\xi)\right]}{q^N}\tag{4.15}$$

which we will do using the Laplace method. Therefore, we first have to find the global minima of $h_\Lambda(\beta, 0, X_\xi)$.

**Lemma 4.5.** Let $s_0(\beta)$ be the function defined in Lemma 3.1 and $\tilde{\beta}=\beta\,(q/(q-1))$. Then, the global minima of $h_\Lambda(\beta, 0, X_\xi)$ are taken on for $X_\xi$ of the form

$$\tilde{X}_\xi^{(\mu)}=\{s_0(\tilde{\beta})\,\varepsilon^{\mu,(\gamma)}\}_{\gamma=1,\dots,q}\tag{4.16}$$

for all $\mu\in\{1,\dots,p\}$. Moreover, we have, for all $\tilde{\beta}\neq\beta_c$ and $\mu\in\{1,\dots,p\}$,

$$\frac{1}{d}h_\Lambda(\beta,0,\tilde{X}_\xi^{(\mu)})=F^{(\mathrm{C.W.})}(\tilde{\beta},h=0)+\frac{\tilde{\beta}}{2q}-\ln q\tag{4.17}$$

with $F^{(\mathrm{C.W.})}$ and $\beta_c$ defined in Section 3. (Note that in the case $q=2$ the condition $\tilde{\beta}\neq\beta_c$ may be dropped.)

*Remark.* Notice that, with the relation between $e^\mu$ and $\xi^\mu$ given by the map $I_\xi$ of Section 2, each minimum is associated to one of the patterns $\xi^\mu$.

*Proof.* Putting $\lambda=0$ in Eq. (4.8), we have

$$h_\Lambda(\beta,0,X_\xi)=\frac{\beta}{2}\left[\left\|\sum_{\gamma=1}^{q}P^{(\gamma)}x^{(\gamma)}\right\|_d^2-\frac{q}{q-1}\sum_{\gamma=1}^{q}\|P^{(\gamma)}x^{(\gamma)}\|_d^2\right]$$
$$+\sum_{k=1}^{d}\sum_{\gamma=1}^{q}\left(x_k^{(\gamma)}+\frac{1}{q}\right)\ln\left(x_k^{(\gamma)}+\frac{1}{q}\right)\tag{4.18}$$

Since

$$\|P^{(\gamma)}x^{(\gamma)}\|_d^2 + \|(\mathbb{1} - P^{(\gamma)})\, x^{(\gamma)}\|_d^2 = \|x^{(\gamma)}\|_d^2$$

and by (4.7)

$$\sum_{\gamma=1}^{q} \left(x_k^{(\gamma)} + \frac{1}{q}\right)^2 = \frac{1}{q} + \sum_{\gamma=1}^{q} (x_k^{(\gamma)})^2$$

we get

$$\begin{aligned} h_A(\beta, 0, X_\xi) = {} & \frac{\beta}{2}\left[\left\| \sum_{\gamma=1}^{q} P^{(\gamma)}x^{(\gamma)}\right\|_d^2 + \frac{q}{q-1} \sum_{\gamma=1}^{q} \|(\mathbb{1} - P^{(\gamma)})\, x^{(\gamma)}\|_d^2\right] \\ & + \sum_{k=1}^{d} \sum_{\gamma=1}^{q} \left\{-\frac{\beta}{2}\left(\frac{q}{q-1}\right)\left(x_k^{(\gamma)} + \frac{1}{q}\right)^2\right. \\ & \left. + \left(x_k^{(\gamma)} + \frac{1}{q}\right) \ln\left[q\left(x_k^{(\gamma)} + \frac{1}{q}\right)\right]\right\} + \frac{\beta d}{2(q-1)} - d \ln q \end{aligned} \tag{4.19}$$

where we recognize from Section 3 the function

$$\sum_{\gamma=1}^{q} \left\{-\frac{\tilde{\beta}}{2}\left(x_k^{(\gamma)} + \frac{1}{q}\right)^2 + \left(x_k^{(\gamma)} + \frac{1}{q}\right) \ln\left[q\left(x_k^{(\gamma)} + \frac{1}{q}\right)\right]\right\} \tag{4.20}$$

which, minimized over the vectors $(x_k^{(1)},..., x_k^{(q)}) \in \mathbb{R}^q$ satisfying the constraint (4.7), gives the free energy $F^{(\text{C.W.})}$ of the Curie–Weiss model for the inverse temperature $\tilde{\beta} = \beta(q/(q-1))$. Since the first two terms in (4.19) are positive, the global minima of $h_A(\beta, 0, X_\xi)$ are attained for all $X_\xi$ that minimize (4.20) for all $k = 1,..., d$ and at the same time annihilate the first two terms. Now $P^{(\gamma)}\varepsilon^{\mu,(\gamma)} = \varepsilon^{\mu,(\gamma)}$ by definition, and taking into account Lemma 2.1, we see that for any vector $x^{(\gamma)}$ of the form $s\varepsilon^{\mu,(\gamma)}$ the first two terms in (4.12) give zero. Moreoever, we have seen in Lemma 3.2 that the term (4.20) is minimized by these vectors exactly if $s$ is chosen as $s_0(\tilde{\beta})$ defined in (3.5), (3.6). We have thus proven the lemma. ∎

We are now ready to prove Proposition 4.3. Let us write

$$S \equiv \sum_{Y_\xi \in \Xi_\xi} \frac{\exp[-(N/d)\, h_A(\beta, 0, X_\xi)]}{q^N}$$

By Lemma 4.5, $S$ satisfies the bounds

$$\begin{aligned} S \geqslant {} & \sum_{\substack{Y_\xi \in \Xi_\xi: \\ X_\xi = \tilde{X}_\xi^{(\mu)}}} \exp\left\{-N\left[F^{(\text{C.W.})}(\tilde{\beta}, h=0) + \frac{\tilde{\beta}}{2q}\right]\right\} \\ = {} & p \exp\left\{-N\left[F^{(\text{C.W.})}(\tilde{\beta}, h=0) + \frac{\tilde{\beta}}{2q}\right]\right\} \end{aligned}$$

and

$$S=\exp\left\{-N\left[F^{(\mathrm{C.W.})}(\tilde{\beta}, h=0)+\frac{\tilde{\beta}}{2q}\right]\right\}$$
$$\times\sum_{Y_\xi\in\Xi_\xi}\exp\left(-N\left\{\frac{1}{d}h_A(\beta, 0, X_\xi)-\left[F^{(\mathrm{C.W.})}(\tilde{\beta}, h=0)+\frac{\tilde{\beta}}{2q}\right]\right\}\right)$$

Using that

$$\exp\left(-N\left\{\frac{1}{d}h_A(\beta, 0, X_\xi)-\left[F^{(\mathrm{C.W.})}(\tilde{\beta}, h=0)+\frac{\tilde{\beta}}{2q}\right]\right\}\right)\leqslant 1$$

for all $Y_\xi\in\Xi_\xi$ and that

$$\sum_{Y_\xi\in\Xi_\xi}1\leqslant\prod_{k=1}^{d}\prod_{\gamma=1}^{q}L_k\leqslant\left[\frac{N}{d}(1+\delta_N)\right]^{qd}\leqslant e^{\delta_N N}\tag{4.21}$$

we get

$$S\leqslant\exp\left\{-N\left[F^{(\mathrm{C.W.})}(\tilde{\beta}, h=0)+\frac{\tilde{\beta}}{2q}\right]\right\}\exp(\delta_N N)$$

Taking logarithms and dividing by $N$, we get

$$\frac{\ln p}{N}-\left\{F^{(\mathrm{C.W.})}(\tilde{\beta}, h=0)+\frac{\tilde{\beta}}{2q}\right\}\leqslant S\leqslant\delta_N$$
$$-\left\{F^{(\mathrm{C.W.})}(\tilde{\beta}, h=0)+\frac{\tilde{\beta}}{2q}\right\}$$

which proves Proposition 4.3, since both $(\ln p)/N$ and $\delta_N$ tend to zero as $N$ tends to infinity. ▮

We finally get the following result.

**Theorem 1.** For all $\tilde{\beta}\neq\beta^c$

$$\lim_{N\to\infty}F_{A,\beta}(\xi)=\tilde{F}(\beta)$$

for almost all $\omega\in\Omega$ and in all $L^r(\Omega, \Sigma, \mathbb{P})$, $1\leqslant r<\infty$, where $\tilde{F}(\beta)$ is defined in Proposition 4.3.

*Proof.* The almost sure convergence is obtained by putting together Proposition 4.1 and Proposition 4.3. The convergence in the spaces

$L^r(\Omega, \Sigma, \mathbb{P})$ follows from the mean convergence criterion,[5] since the random variables $\{|F_\Lambda(\xi)|^r, N \geqslant 1\}$ are uniformly integrable. We give the proof of the uniform integrability in the Appendix.

## 5. STRUCTURE OF THE GIBBS STATES

The form of the free energy functional found in the last section suggests that for $\beta > \beta^c$ the extremal infinite-volume Gibbs state of our model should be the measures $\mathscr{G}^\mu$ concentrated near the original patterns $\xi^\mu$. Of course, taking the limit of $e^{-\beta H}/Z$, we would get a mixed state, namely the symmetric linear combination of the extremal states. To construct the state $\mathscr{G}^\alpha$, we have to add to $H_\Lambda(\sigma; \xi)$ a "magnetic field" coupling to the pattern $\xi^\alpha$, that is, we write

$$H^\alpha_{\Lambda,\eta}(\sigma; \xi) = -\frac{1}{2N} \sum_{\mu=1}^{p} \left\{ \sum_{i=1}^{N} \frac{q}{q-1} \left[ \delta(\xi_i^\mu, \sigma_i) - \frac{1}{q} \right] \right\}^2 - \eta \sum_{i=1}^{N} \left[ \delta(\xi_i^\alpha, \sigma_i) - \frac{1}{q} \right] \tag{5.1}$$

Note that $\eta \geqslant 0$, so that the minimum corresponding to $\xi^\alpha$ is deeper than the other ones. We denote by $\mathscr{G}^\alpha_{\Lambda,\beta,\eta}$ the finite-volume Gibbs measure

$$\mathscr{G}^\alpha_{\Lambda,\beta,\eta}(\sigma; \xi) = \frac{\exp[-\beta H^\alpha_{\Lambda,\eta}(\sigma; \xi)]}{q^N Z^\alpha_{\Lambda,\beta,\eta}(\xi)} \tag{5.2}$$

where

$$Z^\alpha_{\Lambda,\beta,\eta}(\xi) = \sum_{\sigma \in \mathscr{S}^N} \frac{\exp[-\beta H^\alpha_{\Lambda,\eta}(\sigma; \xi)]}{q^N} \tag{5.3}$$

The states $\mathscr{G}^\alpha$ are then obtained by letting first $N \uparrow \infty$ and then $\eta \downarrow 0$.

While we will not actually construct the measures $\mathscr{G}$ explicitly, we construct measures $\mathscr{Q}$ induced by them on the physically interesting "overlap parameters" $m^\mu_\Lambda(\sigma; \xi)$. These are defined as

$$m^\mu_\Lambda(\sigma; \xi) = \frac{1}{N} \sum_{i=1}^{N} \left[ \delta(\xi_i^\mu, \sigma_i) - \frac{1}{q} \right] \tag{5.4}$$

and measure how close a configuration of spins $\sigma$ is to the pattern $\xi^\mu$, in that

$$m^\mu_\Lambda(\sigma; \xi) = \begin{cases} 1 - \dfrac{1}{q} & \text{if} \quad \sigma = \xi^\mu \\ -\dfrac{1}{q} & \text{if} \quad \sigma \text{ is "orthogonal" to } \xi^\mu \end{cases}$$

For a finite-volume Gibbs state $\mathscr{G}^{\alpha}_{\Lambda,\beta,\eta}$, we set

$$m^{\mu,\alpha}_{\Lambda,\beta,\eta}(\xi) = \sum_{\sigma \in \mathscr{S}^N} m^{\mu}_{\Lambda}(\sigma;\xi)\, \mathscr{G}^{\alpha}_{\Lambda,\beta,\eta}(\sigma;\xi) \tag{5.5}$$

Furthermore, for any finite integer $k \leqslant p$ and any family $\{\mu_1,..., \mu_k\} \subset \{1,..., p\}$ we define the map

$$\mathscr{M}^{\mu_1,...,\mu_k}\colon\ \mathscr{S}^N \to \left[-\frac{1}{q}, 1-\frac{1}{q}\right]^k$$

$$\sigma \mapsto (m^{\mu_1}_{\Lambda}(\sigma;\xi),..., m^{\mu_k}_{\Lambda}(\sigma;\xi)) \tag{5.6}$$

Under this map the Gibbs measures $\mathscr{G}^{\alpha}_{\Lambda,\beta,\eta}$ induce measures $\mathscr{Q}^{\alpha}_{\Lambda,\beta,\eta}$ on $[-1/q, 1-1/q]^k$.

We will prove the following theorems:

**Theorem 2.** Let $\tilde{\beta} = [q/(q-1)]\,\beta$ and $s_0(\tilde{\beta})$ defined in Lemma 3.2. Then for all $\tilde{\beta} \neq \beta^c$

$$\lim_{\eta \downarrow 0} \lim_{N \uparrow \infty} m^{\mu,\alpha}_{\Lambda,\beta,\eta}(\xi) = \frac{q-1}{q}\, s_0(\tilde{\beta})\, \delta(\alpha,\mu) \tag{5.7}$$

for almost all $\omega \in \Omega$ and in all $L^r(\Omega, \Sigma, \mathbb{P})$ for $1 \leqslant r < \infty$.

Remember that by Lemma 3.2, $s_0(\tilde{\beta}) = 0$ for $\tilde{\beta} < \beta^c$ and has a jump discontinuity at $\tilde{\beta} = \beta^c$ in the case $q > 2$, which means that the model undergoes a first-order phase transition in the mean overlap order parameter.

**Theorem 3.** Let $\delta^{\alpha}_{\mu_1,...,\mu_k}$ be the Dirac measure in $\mathbb{R}^k$ concentrated on the vector

$$\frac{q-1}{q}\, s_0(\tilde{\beta})(\delta(\mu_1,\alpha),..., \delta(\mu_k,\alpha))$$

Then for all $\tilde{\beta} \neq \beta^c$ and for almost all $\omega \in \Omega$

$$\lim_{\eta \downarrow 0} \lim_{N \uparrow \infty} \mathscr{Q}^{\alpha}_{\Lambda,\beta,\eta} = \delta^{\alpha}_{\mu_1,...,\mu_k}$$

in the weak topology.

*Remark.* Theorem 2 is of course a special case of Theorem 3. We state it separately, since its content is of particular interest. We first prove Theorem 2. The proof of Theorem 3 will be a rather simple extension of that of Theorem 2.

We will finally show that the formalism introduced in the proof of Theorem 2 can be easily generalized to the case where the external magnetic field couples a finite number of patterns with equal strengh. That is, for any finite $l < p$ and any family $\{\alpha_1,..., \alpha_k\} \subset \{1,..., p\}$ we define the Hamiltonian

$$H^{\alpha_1,...,\alpha_l}_{\Lambda,\eta}(\sigma; \xi) = H_{\Lambda,\eta}(\sigma; \xi) - \eta \sum_{j=1}^{l} \sum_{i=1}^{N} \left[ \delta(\xi_i^{\alpha_j}, \sigma_i) - \frac{1}{q} \right]$$

Denoting by $\mathscr{G}^{\alpha_1,...,\alpha_l}_{\Lambda,\beta,\eta}$ and $\mathscr{Q}^{\alpha_1,...,\alpha_l}_{\Lambda,\beta,\eta}$ the associated finite-volume Gibbs measure and its image under the map $\mathscr{M}^{\mu_1,...,\mu_k}$, Theorem 3 becomes as follows.

**Theorem 4.** For all $\tilde{\beta} \neq \beta^c$ and for almost all $\omega \in \Omega$

$$\lim_{\eta \downarrow 0} \lim_{N \uparrow \infty} \mathscr{Q}^{\alpha_1,...,\alpha_l}_{\Lambda,\beta,\eta} = \frac{1}{l} \sum_{j=1}^{l} \delta^{\alpha_j}_{\mu_1,...,\mu_k}$$

in the weak topology.

*Proof of Theorem* 2. We first express the Hamiltonian (5.1) and the overlap parameters in terms of the variables $Y_\xi$. Using the transformations described in Section 2, we find that the effective Hamiltonian associated to (5.1) becomes

$$\bar{H}^{\alpha}_{\Lambda,\eta}(Y_\xi) = \bar{H}_\Lambda(Y_\xi) - \eta \langle \varepsilon^\alpha \mid Y_\xi \rangle_{q,d} \tag{5.8}$$

and for all $\sigma \in \mathscr{S}(Y_\xi)$, the overlap (5.4) takes the value

$$m^\mu_\Lambda(\sigma; \xi) = \frac{1}{N} \langle \varepsilon^\mu \mid Y_\xi \rangle_{q,d} \tag{5.9}$$

where $\langle \varepsilon^\mu \mid Y_\xi \rangle_{q,d} \equiv \sum_{\gamma=1}^{q} \langle \varepsilon^{\mu,(\gamma)} \mid y^{(\gamma)} \rangle_d$. Thus, the mean overlap can be expressed as the expectation with respect to the "canonical" measure $\nu^\alpha_{\Lambda,\beta,\eta}$ associated to the effective Hamiltonian, that is,

$$m^{\mu,\alpha}_{\Lambda,\beta,\eta}(\xi) = \sum_{Y_\xi \in \Xi_\xi} \frac{1}{N} \langle \varepsilon^\mu \mid Y_\xi \rangle_{q,d} \, \nu^\alpha_{\Lambda,\beta,\eta}(Y_\xi) \tag{5.10}$$

where

$$\nu^\alpha_{\Lambda,\beta,\eta}(Y_\xi) = \frac{\exp[-\beta \bar{H}^\alpha_{\Lambda,\eta}(Y_\xi)]}{q^N Z^\alpha_{\Lambda,\beta,\eta}(\xi)} \tag{5.11}$$

and

$$Z^\alpha_{\Lambda,\beta,\eta}(\xi) = \sum_{Y_\xi \in \Xi_\xi} \frac{\exp[-\beta \bar{H}^\alpha_{\Lambda,\eta}(Y_\xi)]}{q^N} \tag{5.12}$$

In order to prove the theorem, we need a generalization of Lemma 4.4 for nonzero magnetic field. Let us define

$$h_\Lambda^\alpha(\beta, \lambda, \eta, X_\xi) = h_\Lambda(\beta, \lambda, X_\xi) - \beta\eta \sum_{\bar{y}\gamma=1}^{q} \langle \varepsilon^{\alpha,(\gamma)} | (\mathbb{1} + M) x^{(\gamma)} \rangle_d \tag{5.13}$$

Then we have the following result.

**Lemma 5.1.** For all $\omega$ in $\tilde{\Omega}$ there exists $N_0$ such that, for all $N > N_0$,

$$\left| \beta \bar{H}_{\Lambda,\eta}^\alpha(Y_\xi) - \frac{N}{d} h_\Lambda^\alpha(\beta, \lambda, \eta, X_\xi)|_{\lambda=0} \right| \leqslant \left[ c(\beta, q) + \beta q \left( \frac{q-1}{q} \right)^2 \eta \right] \delta_N N \tag{5.14}$$

for some constant $c(\beta, q)$ independent of $N$ and $p$.

*Proof.* This lemma is proven exactly as Lemma 4.4 Just notice that we have the bound

$$\left| \sum_{\gamma=1}^{q} \langle \varepsilon^{\alpha,(\gamma)} | M x^{(\gamma)} \rangle_d \right| \leqslant qd \left( \frac{q-1}{q} \right)^2 \delta_N \quad \blacksquare \tag{5.15}$$

Remember that $\tilde{X}_\xi^{(\mu)} = \{ s_0(\tilde{\beta})\, \varepsilon^{\mu,(\gamma)} \}_{\gamma=1,\dots,q}$, $\mu = 1,\dots, p$, are the minima of $h_\Lambda(\beta, 0, X_\xi)$. We denote by $X_\xi^*$ the *unique* minimum of $h_\Lambda^\alpha(\beta, \lambda, \eta, X_\xi)$ for nonzero $\eta$. We put

$$\tilde{m}_{\Lambda,\beta,\eta}^{\mu,\alpha}(\xi) = \frac{1}{d} \langle \varepsilon^\mu | X_\xi^* \rangle_{q,d} \tag{5.16}$$

for $\mu = 1,\dots, p$. Now

$$\begin{aligned} \frac{1}{N} \langle \varepsilon^\mu | Y_\xi \rangle_{q,d} &= \frac{1}{d} \sum_{\gamma=1}^{q} \langle \varepsilon^{\mu,(\gamma)} | (\mathbb{1} + M) x^{(\gamma)} \rangle_d \\ &= \frac{1}{d} \sum_{\gamma=1}^{q} \langle \varepsilon^{\mu,(\gamma)} | (\mathbb{1} + M) x^{(\gamma)} - x^{*,(\gamma)} \rangle_d + \tilde{m}_{\Lambda,\beta,\eta}^{\mu,\alpha}(\xi) \\ &= \frac{1}{d} \sum_{\gamma=1}^{q} \langle \varepsilon^{\mu,(\gamma)} | M x^{(\gamma)} \rangle_d + \frac{1}{d} \langle \varepsilon^\mu | X_\xi - X_\xi^* \rangle_{q,d} + \tilde{m}_{\Lambda,\beta,\eta}^{\mu,\alpha}(\xi) \end{aligned} \tag{5.17}$$

Using (5.15), we thus have

$$\begin{aligned} &|m_{\Lambda,\beta,\eta}^{\mu,\alpha}(\xi) - \tilde{m}_{\Lambda,\beta,\eta}^{\mu,\alpha}(\xi)| \\ &\quad \leqslant q \left( \frac{q-1}{q} \right)^2 \delta_N + \left| \sum_{Y_\xi \in \Xi_\xi} \frac{1}{d} \langle \varepsilon^\mu | X_\xi - X_\xi^* \rangle_{q,d}\, \nu_{\Lambda,\beta,\eta,\alpha}(Y_\xi) \right| \end{aligned} \tag{5.18}$$

Our strategy will be to prove that the right-hand side of (5.18) converges to zero as $N$ tends to infinity for $\eta$ small enough provided $\omega$ is in the space $\tilde{\Omega}$. To do so, we decompose the sum over $Y_\xi \in \Xi_\xi$ into two pieces. We define the set

$$A = \{ Y_\xi \in \Xi_\xi : \|X_\xi - X_\xi^*\|_{q,d} \leqslant \sqrt{d}\, \rho(\eta, N) \} \tag{5.19}$$

where $\rho$ is a function of $\eta$ and $N$ such that $\rho \downarrow 0$ as $N \uparrow \infty$ and will be chosen appropriately later. Using that (see Lemma 2.1)

$$|\langle \varepsilon^\mu | X_\xi - X_\xi^* \rangle_{q,d}| \leqslant \|\varepsilon^\mu\|_{q,d} \, \|X_\xi - X_\xi^*\|_{q,d} \leqslant d\rho(\eta, N) \left(\frac{q-1}{q}\right)^{1/2} \tag{5.20}$$

this definition implies the bound

$$|m^{\mu,\alpha}_{\Lambda,\beta,\eta}(\xi) - \tilde{m}^{\mu,\alpha}_{\Lambda,\beta,\eta}(\xi)| \leqslant q \left(\frac{q-1}{q}\right)^2 \delta_N + \left(\frac{q-1}{q}\right)^{1/2} \rho(\eta, N) + \left| \sum_{Y_\xi \in A^c} \frac{1}{d} \langle \varepsilon^\mu | X_\xi - X_\xi^* \rangle_{q,d} \, v^\alpha_{\Lambda,\beta,\eta}(Y_\xi) \right| \tag{5.21}$$

The first two terms in (5.21) vanish by definition as $N \uparrow \infty$, so that we are only interested in the third. Note that we have a uniform bound

$$\frac{1}{d} |\langle \varepsilon^\mu | X_\xi - X_\xi^* \rangle_{q,d}| \leqslant 2q \left(\frac{q-1}{q}\right)^2 \tag{5.22}$$

so that we are left to study $v^\alpha_{\Lambda,\beta,\eta}(Y_\xi)$. Obviously,

$$v^\alpha_{\Lambda,\beta,\eta}(Y_\xi) = \left( \exp\left\{ -\left[ \beta \bar{H}^\alpha_{\Lambda,\eta}(Y_\xi) - \frac{N}{d} h^\alpha_\Lambda(\beta, 0, \eta, X_\xi) \right] - \frac{N}{d} h^\alpha_\Lambda(\beta, 0, \eta, X_\xi) \right\} \right) \times \left( \sum_{Y_\xi \in \Xi_\xi} \exp\left\{ -\left[ \beta \bar{H}^\alpha_{\Lambda,\eta}(Y_\xi) - \frac{N}{d} h^\alpha_\Lambda(\beta, 0, \eta, X_\xi) \right] - \frac{N}{d} h^\alpha_\Lambda(\beta, 0, \eta, X_\xi) \right\} \right)^{-1}$$

and using Lemma 5.1,

$$\begin{aligned} &\leqslant \exp(2\delta_N N) \frac{\exp[-(N/d)\, h^\alpha_\Lambda(\beta, 0, \eta, X_\xi)]}{\sum_{Y_\xi \in \Xi_\xi} \exp[-(N/d)\, h^\alpha_\Lambda(\beta, 0, \eta, X_\xi)]} \\ &= \exp(2\delta_N N) \frac{\exp\{-(N/d)[h^\alpha_\Lambda(\beta, 0, \eta, X_\xi) - h^\alpha_\Lambda(\beta, 0, \eta, X_\xi^*)]\}}{\sum_{Y_\xi \in \Xi_\xi} \exp\{-(N/d)[h^\alpha_\Lambda(\beta, 0, \eta, X_\xi) - h^\alpha_\Lambda(\beta, 0, \eta, X_\xi^*)]\}} \end{aligned} \tag{5.23}$$

Since the denominator in (5.23) is larger than one, we arrive at

$$\nu^{\alpha}_{\Lambda,\beta,\eta}(Y_{\xi}) \leqslant \exp(2\delta_N N) \exp\left\{-\frac{N}{d}\left[h^{\alpha}_{\Lambda}(\beta, 0, \eta, X_{\xi}) - h^{\alpha}_{\Lambda}(\beta, 0, \eta, X^*_{\xi})\right]\right\}$$

Therefore

$$\begin{aligned} |m^{\mu,\alpha}_{\Lambda,\beta,\eta}(\xi) - \tilde{m}^{\mu,\alpha}_{\Lambda,\beta,\eta}(\xi)| &\leqslant q\left(\frac{q-1}{q}\right)^2 \delta_N + \left(\frac{q-1}{q}\right)^{1/2} \rho(\eta, N) \\ &\quad + 2q\left(\frac{q-1}{q}\right)^2 \exp(2\delta_N N) \\ &\quad \times \sum_{Y_{\xi}\in A^c} \exp\left\{-\frac{N}{d}\left[h^{\alpha}_{\Lambda}(\beta, 0, \eta, X_{\xi}) - h^{\alpha}_{\Lambda}(\beta, 0, \eta, X^*_{\xi})\right]\right\} \end{aligned} \tag{5.24}$$

To bound the exponential in the last equation, we must introduce some technicalities. Let us write $x_k$ for the vector in $\mathbb{R}^q$ whose components are given by $x_k^{(\gamma)}$. Notice that for $\eta$ small the absolute minimum $X^*_{\xi}$ will be a small shift of the corresponding minimum $\tilde{X}^{(\alpha)}_{\xi}$ in zero field, that is, for all $\gamma = 1,..., q$,

$$x^{*,(\gamma)} = \tilde{x}^{\alpha,(\gamma)} + \varepsilon^{\alpha,(\gamma)}\vartheta^{(\gamma)}(\eta) \tag{5.25}$$

Since $h_{\Lambda}(\beta, \lambda, X_{\xi})$ is a sum of $d$ real analytic functions symmetric with respect to the index $\gamma$, it follows by the perturbation theory that $\vartheta(\eta)$ is independent of $\gamma$ and $\vartheta(\eta) = O(\eta)$ if $\eta$ is small enough. Remember that the $x_k$ take value in the subset

$$\Delta_q = \left\{x_k \in \mathbb{R}^q \;\middle|\; \sum_{\gamma=1}^{q} x_k^{(\gamma)} + \frac{1}{q} = 1,\ x_k^{(\gamma)} + \frac{1}{q} \geqslant 0\right\}$$

We will divide $\Delta_q$ into $q$ pieces $\Delta_q^{\gamma}$ defined as

$$\Delta_q^{\gamma} = \{x_k \in \Delta_q \mid x_k^{(\gamma)} \geqslant x_k^{(\gamma')} \text{ for all } \gamma' = 1,..., q; \gamma' \neq \gamma\} \tag{5.26}$$

It is easy to check that $\Delta_q^{\gamma}$ is in fact the subset of points $x_k \in \Delta_q$ for which $\tilde{z}^{(\gamma)}$ is the closest of the minima $\tilde{z}^{(1)},..., \tilde{z}^{(q)}$ defined at the end of Section 3. Let $x_k \in \Delta_q^{\gamma}$ and $x_k^* \in \Delta_q^{\gamma^*}$. We then denote by $\mathscr{T}_{x_k}$ the reflection on a hyperplane that takes $\Delta_q^{\gamma^*}$ into $\Delta_q^{\gamma}$. We will denote by $\mathscr{T}_{X_{\xi}}(X')$ the ensemble of the $d$ vectors $\{\mathscr{T}_{x_k}(x'_k)\}_{k=1,...,d}$. With this relation we can now announce the following result.

**Lemma 5.2.** There exists a constant $a > 0$, depending only on $q$, such that for all $Y_\xi \in \Xi_\xi$ and for $\eta$ sufficiently small

$$h^\alpha_\Lambda(\beta, 0, \eta, X_\xi) - h^\alpha_\Lambda(\beta, 0, \eta, X^*_\xi) \geqslant a \, \|X_\xi - \mathscr{T}_{X_\xi}(X^*_\xi)\|^2_{q,d} + \beta\eta \langle \mathscr{T}_{X_\xi}(\varepsilon^\alpha) - \varepsilon^\alpha \mid X_\xi \rangle_{q,d} \quad (5.27)$$

where $\langle \mathscr{T}_{X_\xi}(\varepsilon^\alpha) - \varepsilon^\alpha \mid X_\xi \rangle_{q,d} \geqslant 0$.

*Proof.* Let us write

$$g(x_k) = -\sum_{\gamma=1}^{q} \left\{ \frac{\tilde{\beta}}{2}\left(x^{(\gamma)}_k + \frac{1}{q}\right)^2 + \left(x^{(\gamma)}_k + \frac{1}{q}\right) \ln\left[q\left(x^{(\gamma)}_k + \frac{1}{q}\right)\right]\right\}$$

$$f_k(x_k) = -\beta\eta \langle \varepsilon^\alpha_k \mid x_k \rangle_q$$

that is,

$$h^\alpha_\Lambda(\beta, 0, \eta, X_\xi) = \frac{\beta}{2}\left[\left\| \sum_{\gamma=1}^{q} P^{(\gamma)} x^{(\gamma)} \right\|^2_d + \frac{q}{q-1} \sum_{\gamma=1}^{q} \|(\mathbb{1} - P^{(\gamma)})\, x^{(\gamma)}\|^2_d\right] + \sum_{k=1}^{d} (g + f_k)(x_k) + \frac{\beta d}{2(q-1)} - d \ln q$$

We may write

$$(g + f_k)(x_k) = (g + f_k)(x_k) - (g + f_k)[\mathscr{T}_{x_k}(x^*_k)] + (g + f_k)[\mathscr{T}_{x_k}(x^*_k)]$$

Furthermore,

$$f_k(\mathscr{T}_{x_k}(x^*_k)) = -\beta\eta \langle \mathscr{T}_{x_k}(\varepsilon^\alpha_k) \mid \mathscr{T}_{x_k}(x^*_k) \rangle_q + \beta\eta \langle \mathscr{T}_{x_k}(\varepsilon^\alpha_k) - \varepsilon^\alpha_k \mid \mathscr{T}_{x_k}(x^*_k) \rangle_q$$

and

$$f_k(x_k) = -\beta\eta \langle \varepsilon^\alpha_k \mid \mathscr{T}_{x_k}(x_k) \rangle_q + \beta\eta \langle \mathscr{T}_{x_k}(\varepsilon^\alpha_k) - \varepsilon^\alpha_k \mid x_k \rangle_q$$

Combining these equations, we get

$$\begin{aligned}(g + f_k)(x_k) &= [\{g(x_k) - \beta\eta \langle \mathscr{T}_{x_k}(\varepsilon^\alpha_k) \mid x_k \rangle_q\} - \{g(\mathscr{T}_{x_k}(x^*_k)) \\ &\quad - \beta\eta \langle \mathscr{T}_{x_k}(\varepsilon^\alpha_k) \mid \mathscr{T}_{x_k}(x^*_k) \rangle_q\}] + \{g(\mathscr{T}_{x_k}(x^*_k)) \\ &\quad - \beta\eta \langle \mathscr{T}_{x_k}(\varepsilon^\alpha_k) \mid \mathscr{T}_{x_k}(x^*_k) \rangle_q\} + \beta\eta \langle \mathscr{T}_{x_k}(\varepsilon^\alpha_k) - \varepsilon^\alpha_k \mid x_k \rangle_q \\ &= [1] + \{2\} + \langle 3 \rangle\end{aligned}$$

Since $x_k$ takes values in a compact set and $\mathscr{T}_{x_k}(x^*_k)$ realizes the minimum of the function

$$g(x_k) - \beta\eta \langle \mathscr{T}_{x_k}(\varepsilon^\alpha_k) \mid x_k \rangle_q$$

there exists a constant $a>0$ such that the term [1] is bounded by

$$[1] \geqslant a \, \|x_k - \mathscr{T}_{x_k}(x_k^*)\|_q^2$$

To deal with the term {2}, note that $g$ is invariant under permutations of the components of a vector $x_k$ and that

$$\langle \mathscr{T}_{x_k}(\varepsilon_k^\alpha) \mid \mathscr{T}_{x_k}(x_k^*) \rangle_q = \langle \varepsilon_k^\alpha \mid x_k^* \rangle_q$$

Therefore

$$\{2\} = (g+f_k)(x_k^*)$$

Finally,

$$(g+f_k)(x_k) \geqslant a \, \|x_k - \mathscr{T}_{x_k}(x_k^*)\|_q^2 + (g+f_k)(x_k^*) + \beta\eta \langle \mathscr{T}_{x_k}(\varepsilon_k^\alpha) - \varepsilon_k^\alpha \mid x_k \rangle_q$$

It is easy to check that the term $\langle \mathscr{T}_{x_k}(\varepsilon_k^\alpha) - \varepsilon_k^\alpha \mid x_k \rangle_q$ is positive. To do so, let us suppose that $x_k \in \Delta_q^{\gamma'}$ and $\varepsilon_k^\alpha \in \Delta_q^{\gamma''}$, then $\mathscr{T}_{x_k}(\varepsilon_k^\alpha) \in \Delta_q^{\gamma'}$. Using the constraint $\sum_{\gamma=1}^q x_k^\gamma = 0$, we get $\langle \mathscr{T}_{x_k}(\varepsilon_k^\alpha) \mid x_k \rangle_q = x_k^{\gamma'}$, $\langle \varepsilon_k^\alpha \mid x_k \rangle_q = x_k^{\gamma''}$, and $x_k^{\gamma'} \geqslant x_k^{\gamma''}$, since $x_k \in \Delta_q^{\gamma'}$.

The lemma is now obtained by realizing that the term

$$\left\| \sum_{\gamma=1}^q P^{(\gamma)} x^{(\gamma)} \right\|_d^2 + \frac{q}{q-1} \sum_{\gamma=1}^q \|(\mathbb{1} - P^{(\gamma)}) \, x^{(\gamma)}\|_d^2$$

is positive and vanishes at a point $X_\xi^*$. ∎

We will also make use of the following result.

**Lemma 5.3.** For all $Y_\xi \in \Xi_\xi$ and $X_\xi^*$ defined in (5.25)

$$\|\mathscr{T}_{X_\xi}(X_\xi^*) - X_\xi^*\|_{q,d}^2 = 2 \langle \mathscr{T}_{X_\xi}(X_\xi^*) - X_\xi^* \mid \mathscr{T}_{X_\xi}(X_\xi^*) \rangle_{q,d} \tag{5.28}$$

*Proof.* Since

$$\|\mathscr{T}_{X_\xi}(X_\xi^*) - X_\xi^*\|_{q,d}^2 = \{s_0(\beta)[1+O(\eta)]\}^2 \sum_{k=1}^d \|\mathscr{T}_{x_k}(\varepsilon_k^\alpha) - \varepsilon_k^\alpha\|_q^2$$

and

$$\langle \mathscr{T}_{X_\xi}(X_\xi^*) - X_\xi^* \mid \mathscr{T}_{X_\xi}(X_\xi^*) \rangle_{q,d} = \{s_0(\beta)[1+O(\eta)]\}^2 \times \sum_{k=1}^d \langle \mathscr{T}_{x_k}(\varepsilon_k^\alpha) - \varepsilon_k^\alpha \mid \mathscr{T}_{x_k}(\varepsilon_k^\alpha) \rangle_q$$

the lemma follows from the equality

$$\|\mathscr{T}_{x_k}(\varepsilon_k^\alpha)-\varepsilon_k^\alpha\|_q^2=2\langle\mathscr{T}_{x_k}(\varepsilon_k^\alpha)-\varepsilon_k^\alpha\mid\mathscr{T}_{x_k}(\varepsilon_k^\alpha)\rangle_q \tag{5.29}$$

which only uses the fact that $\mathscr{T}$ is a reflection. ∎

We are now ready to prove the following.

**Proposition 5.4.** There exists $\eta_0>0$ such that, for all $0<\eta<\eta_0$, and $\tilde{\beta}\neq\beta^c$,

$$\lim_{N\to\infty}\chi_{[\omega\in\tilde{\Omega}]}m^{\mu,\alpha}_{\Lambda,\beta,\eta}(\xi)=\chi_{[\omega\in\tilde{\Omega}]}\tilde{m}^{\mu,\alpha}_{\Lambda,\beta,\eta}(\xi)$$

*Proof.* Let us define the set

$$B=\{Y_\xi\in\Xi_\xi:\ \|X_\xi-\mathscr{T}_{X_\xi}(X_\xi^*)\|_{q,d}\geqslant\sqrt{d}\,\tilde{\rho}(\eta,N)\} \tag{5.30}$$

where $\tilde{\rho}$ is a function of $\eta$ and $N$ such that $\tilde{\rho}\downarrow 0$ as $N\uparrow\infty$ and will have to be chosen suitably. Then the sum over $Y_\xi\in A^c$ in (5.24) can be decomposed into two pieces, either $Y_\xi\in A^c\cap B$ or $Y_\xi\in A^c\cap B^c$. For $Y_\xi A^c\cap B$ we have that

$$\begin{aligned}\exp(2\delta_N N)&\sum_{Y_\xi\in A^c\cap B}\exp\left\{-\frac{N}{d}\left[h_\Lambda^\alpha(\beta,0,\eta,X_\xi)-h_\Lambda^\alpha(\beta,0,\eta,X_\xi^*)\right]\right\}\\ &\leqslant\exp(2\delta_N N)\exp(\delta_N N)\exp\left\{-\frac{N}{d}\left[ad\tilde{\rho}^2(\eta,N)\right]\right\}\\ &=\exp\{[3\delta_N-a\tilde{\rho}^2(\eta,N)]\,N\}\end{aligned} \tag{5.31}$$

where we have used (4.21). The last expression converges to zero as $N\uparrow\infty$, provided only $\tilde{\rho}(\eta,N)$ is chosen in such a way that $a\tilde{\rho}^2(\eta,N)>3\delta_N$, which can be done easily for any $a$.

On the other hand, to bound the sum over $Y_\xi\in A^c\cap B^c$ we will use that the positive term $\langle\mathscr{T}_{X_\xi}(\varepsilon^\alpha)-\varepsilon^\alpha\mid X_\xi\rangle_{q,d}$ should be large enough, since $X_\xi$ is far from $X_\xi^*$. By (5.25) we rewrite this term as

$$\begin{aligned}&\langle\mathscr{T}_{X_\xi}(\varepsilon^\alpha)-\varepsilon^\alpha\mid X_\xi\rangle_{q,d}\\ &\quad=b\langle\mathscr{T}_{X_\xi}(X_\xi^*)-X_\xi^*\mid X_\xi\rangle_{q,d}\\ &\quad=b[\langle\mathscr{T}_{X_\xi}(X_\xi^*)-X_\xi^*\mid X_\xi-\mathscr{T}_{X_\xi}(X_\xi^*)\rangle_{q,d}\\ &\qquad+\langle\mathscr{T}_{X_\xi}(X_\xi^*)-X_\xi^*\mid\mathscr{T}_{X_\xi}(X_\xi^*)\rangle_{q,d}]\end{aligned} \tag{5.32}$$

where we have put $b=1/\{s_o(\beta)[1+O(\eta)]\}$. Remembering that $Y_\xi\in A^c\cap B^c$ and making use of the triangle inequality

$$\|X_\xi-\mathscr{T}_{X_\xi}(X_\xi^*)\|_{q,d}+\|\mathscr{T}_{X_\xi}(X_\xi^*)-X_\xi^*\|_{q,d}\geqslant\|X_\xi-X_\xi^*\|_{q,d}$$

we get the bound

$$\|\mathcal{T}_{X_\xi}(X_\xi^*) - X_\xi^*\|_{q,d} \geqslant [\rho(\eta, N) - \tilde{\rho}(\eta, N)]\sqrt{d} \tag{5.33}$$

which together with Lemma 5.3 yields

$$\langle \mathcal{T}_{X_\xi}(X_\xi^*) - X_\xi^* \mid \mathcal{T}_{X_\xi}(X_\xi^*)\rangle_{q,d} \geqslant \frac{d}{2}[\rho(\eta, N) - \tilde{\rho}(\eta, N)]^2 \tag{5.34}$$

Now we have

$$\begin{aligned} &|\langle \mathcal{T}_{X_\xi}(X_\xi^*) - X_\xi^* \mid X_\xi - \mathcal{T}_{X_\xi}(X_\xi^*)\rangle_{q,d}| \\ &\quad \leqslant \|\mathcal{T}_{X_\xi}(X_\xi^*) - X_\xi^*\|_{q,d}\, \|X_\xi - \mathcal{T}_{X_\xi}(X_\xi^*)\|_{q,d} \\ &\quad = \left(\frac{q-1}{q}\right) d\tilde{\rho}(\eta, N) \end{aligned} \tag{5.35}$$

Thus, the first term in (5.32) is bounded by

$$\langle \mathcal{T}_{X_\xi}(X_\xi^*) - X_\xi^* \mid X_\xi - \mathcal{T}_{X_\xi}(X_\xi^*)\rangle_{q,d} \geqslant -\left(\frac{q-1}{q}\right) d\tilde{\rho}(\eta, N) \tag{5.36}$$

and we finally get

$$\begin{aligned} \langle \mathcal{T}_{X_\xi}(\varepsilon^\alpha) - \varepsilon^\alpha \mid X_\xi\rangle_{q,d} \geqslant b\Bigg\{ &-\left(\frac{q-1}{q}\right) d\tilde{\rho}(\eta, N) \\ &+ \frac{d}{2}[\rho(\eta, N) - \tilde{\rho}(\eta, N)]^2\Bigg\} \end{aligned} \tag{5.37}$$

Therefore

$$\begin{aligned} &\exp(2\delta_N N) \sum_{Y_\xi \in A^c \cap B^c} \exp\left\{-\frac{N}{d}[h_\Lambda^\alpha(\beta, 0, \eta, X_\xi) - h_\Lambda^\alpha(\beta, 0, \eta, X_\xi^*)]\right\} \\ &\quad \leqslant \exp(2\delta_N N)\exp\Bigg\{-\frac{N}{d} b\beta\eta\Bigg[-\left(\frac{q-1}{q}\right) d\tilde{\rho}(\eta, N) \\ &\quad\quad + \frac{d}{2}[\rho(\eta, N) - \tilde{\rho}(\eta, N)]^2\Bigg]\Bigg\} \\ &\quad \leqslant \exp\left\{N\left[3\delta_N + b\beta\eta\left(\frac{q-1}{q}\right)\tilde{\rho}(\eta, N) - \frac{b\beta\eta}{2}[\rho(\eta, N) - \tilde{\rho}(\eta, N)]^2\right]\right\} \end{aligned} \tag{5.38}$$

which again converges to zero for $\tilde{\rho}$ satisfying $a\tilde{\rho}^2(\eta, N) > 3\delta_N$ and for any fixed $\eta > 0$ provided $\rho(\eta, N)$ is chosen such that

$$\frac{b\beta\eta}{2}[\rho(\eta, N) - \tilde{\rho}(\eta, N)]^2 > 3\delta_N + b\beta\eta\left(\frac{q-1}{q}\right)\tilde{\rho}(\eta, N) \quad \blacksquare \tag{5.39}$$

Notice that in the case $\tilde{\beta} < \beta^c$ the above result can be proven in simpler way, since the minimum $X_\xi^*$ is now just a small displacement of the unique minimum in zero field.

By Propositions 5.4 and 5.5, the proof of Theorem 2 is now immediate. Just note that by Proposition 4.1, $\chi_{[\omega \in \tilde{\Omega}]} = 1$ almost surely. Moreover, for all $\tilde{\beta} > \beta^c$, $X_\xi^*$ converges to $s_0(\tilde{\beta})\,\varepsilon^\alpha$ as $\eta$ goes to zero and thus

$$\lim_{\eta \to 0} \tilde{m}^{\mu,\alpha}_{\Lambda,\beta,\eta}(\xi) = \frac{s_0(\tilde{\beta})}{d}\langle \varepsilon^\mu \mid \varepsilon^\alpha \rangle_{q,d} = \frac{q-1}{q} s_o(\tilde{\beta})\,\delta(\mu, \alpha)$$

The convergence in mean of order $r$ follows from the bounded convergence theorem since $|m^{\mu,\alpha}_{\Lambda,\beta,\eta}(\xi)| < 1$. $\blacksquare$

*Proof of Theorem 3.* Since $[-1/q, 1-1/q]^{\mathbb{N}}$ is a compact space, using the Weierstrass theorem, it is enough to prove that for any family of continuous bounded functions $f_1,..., f_k \in \mathscr{C}^b([-1/q, 1-1/q]^{\mathbb{N}}, \mathbb{R})$

$$\begin{aligned} &\lim_{\eta \downarrow 0} \lim_{N \uparrow \infty} \mathscr{Q}^\alpha_{\Lambda,\beta,\eta}(f_1(m^{\mu_1}_\Lambda(\sigma;\xi)) \times \cdots \times f_k(m^{\mu_k}_\Lambda(\sigma;\xi))\,\chi_{[\omega \in \tilde{\Omega}]} \\ &\quad = f_1\left(\frac{q-1}{q} s_0(\tilde{\beta})\,\delta(\alpha, \mu_1)\right) \times \cdots \times f_k\left(\frac{q-1}{q} s_0(\tilde{\beta})\,\delta(\alpha, \mu_k)\right)\chi_{[\omega \in \tilde{\Omega}]} \end{aligned} \tag{5.40}$$

which obviously follows from the results established to prove Theorem 2. To see this fact, notice that

$$\begin{aligned} &\left|\prod_{i=1}^k f_i(m^{\mu_i}_\Lambda(\sigma;\xi)) - \prod_{i=1}^k f_i(\tilde{m}^{\mu_i,\alpha}_{\Lambda,\beta,\eta}(\xi))\right| \\ &\quad = \left|\sum_{i=1}^k \left\{\prod_{j=1}^{i-1} f_j(\tilde{m}^{\mu_j,\alpha}_{\Lambda,\beta,\eta}(\xi))[f_i(m^{\mu_i}_\Lambda(\sigma;\xi)) - f_i(\tilde{m}^{\mu_i,\alpha}_{\Lambda,\beta,\eta}(\xi))]\right.\right. \\ &\qquad \left.\left. \times \prod_{j=i+1}^k f_j(m^{\mu_j}_\Lambda(\sigma;\xi))\right\}\right| \end{aligned} \tag{5.41}$$

where $\tilde{m}^{\mu_i,\alpha}_{\Lambda,\beta,\eta}$ is defined in (5.16). Since $f_j$ are bounded functions, there exists a constant $c$ such that $|f_j| < c$ for all $j = 1,..., \kappa$ and hence (5.41) is bounded by

$$\sum_{i=1}^k c^{k-1}\,|f_i(m^{\mu_i}_\Lambda(\sigma;\xi)) - f_i(\tilde{m}^{\mu_i,\alpha}_{\Lambda,\beta,\eta}(\xi))| \tag{5.42}$$

Thus, for $\omega \in \tilde{\Omega}$, we are left to study the quantity

$$\mathscr{Q}^{\alpha}_{\Lambda,\beta,\eta}\left(\sum_{i=1}^{k} c^{k-1}\,|f_i(m^{\mu_i}_{\Lambda}(\sigma;\xi)) - f_i(\tilde{m}^{\mu_i,\alpha}_{\Lambda,\beta,\eta}(\xi))|\right)$$
$$= c^{k-1}\sum_{i=1}^{k}\sum_{Y_\xi \in \Xi_\xi}\left|f_i\left(\frac{1}{N}\langle \varepsilon^{\mu_i} | Y_\xi\rangle_{q,d}\right) - f_i\left(\frac{1}{d}\langle \varepsilon^{\mu_i} | X^*_\xi\rangle_{q,d}\right)\right| \nu^{\alpha}_{\Lambda,\beta,\eta}(Y_\xi) \tag{5.43}$$

which we will do again by decomposing the last sum into the sums over $Y_\xi \in A$ and $Y_\xi \in A^c$. In the case $Y_\xi \in A$ we have by (5.17) and (5.15)

$$\left|\frac{1}{N}\langle \varepsilon^{\mu_i} | Y_\xi\rangle_{q,d} - \frac{1}{d}\langle \varepsilon^{\mu_i} | X^*_\xi\rangle_{q,d}\right| \leqslant q\left(\frac{q-1}{q}\right)^2 \delta_N + \left(\frac{q-1}{q}\right)^{1/2} \rho(\eta, N) \tag{5.44}$$

and by continuity

$$\left|f_i\left(\frac{1}{N}\langle \varepsilon^{\mu_i} | Y_\xi\rangle_{q,d}\right) - f_i\left(\frac{1}{d}\langle \varepsilon^{\mu_i} | X^*_\xi\rangle_{q,d}\right)\right| < \varepsilon \tag{5.45}$$

for any arbitrarily small $\varepsilon$ provided that $N$ is sufficiently small. Thus,

$$\sum_{Y_\xi \in A}\left|f_i\left(\frac{1}{N}\langle \varepsilon^{\mu_i} | Y_\xi\rangle_{q,d}\right) - f_i\left(\frac{1}{d}\langle \varepsilon^{\mu_i} | X^*_\xi\rangle_{q,d}\right)\right| \nu^{\alpha}_{\Lambda,\beta,\eta}(Y_\xi) \leqslant \varepsilon \tag{5.46}$$

On the other hand, for $Y_\xi \in A^c$ and using the uniform bound

$$\left|f_i\left(\frac{1}{N}\langle \varepsilon^{\mu_i} | Y_\xi\rangle_{q,d}\right) - f_i\left(\frac{1}{d}\langle \varepsilon^{\mu_i} | X^*_\xi\rangle_{q,d}\right)\right| < 2c \tag{5.47}$$

we get

$$\sum_{Y_\xi \in A^c}\left|f_i\left(\frac{1}{N}\langle \varepsilon^{\mu_i} | Y_\xi\rangle_{q,d}\right) - f_i\left(\frac{1}{d}\langle \varepsilon^{\mu_i} | X^*_\xi\rangle_{q,d}\right)\right| \nu^{\alpha}_{\Lambda,\beta,\eta}(Y_\xi) < 2c \sum_{Y_\xi \in A^c} \nu^{\alpha}_{\Lambda,\beta,\eta}(Y_\xi) \tag{5.48}$$

and we have already shown that the right-hand side of (5.48) converges to zero as $N \uparrow \infty$ for any small $\eta > 0$. ∎

*Proof of Theorem 4.* We will substitute in the notations the index $\alpha$ by the family $\alpha_1,..., \alpha_l$ for all the quantities associated to $H_{\Lambda,\eta}^{\alpha_1,...,\alpha_l}$. Under the same assumptions as in the proof of Theorem 3, it is enough to show that

$$\lim_{\eta \downarrow 0} \lim_{N \uparrow \infty} \mathcal{Q}_{\Lambda,\beta,\eta}^{\alpha_1,...,\alpha_l}\left( \prod_{i=1}^{k} f_i(m_\Lambda^{\mu_i}(\sigma;\xi))\right) \chi_{[\omega \in \tilde{\Omega}]}$$
$$= \frac{1}{l} \sum_{j=1}^{l} \prod_{i=1}^{k} f_i\left(\frac{q-1}{q} s_o(\tilde{\beta})\, \delta(\alpha_j, \mu_i)\right) \chi_{[\omega \in \tilde{\Omega}]} \tag{5.49}$$

Denoting by $X_\xi^{*,(\alpha_1)},..., X_\xi^{*,(\alpha_l)}$ the $l$ absolute minima of $\bar{H}_{\Lambda,\eta}^{\alpha_1,...,\alpha_l}$ and by $A^{(\alpha_1)},..., A^{(\alpha_l)}$ the subsets

$$A^{(\alpha_j)} = \{ Y_\xi \in \Xi_\xi : \|X_\xi - X_\xi^{*,(\alpha_j)}\|_{q,d} \leqslant \sqrt{d}\, \rho(\eta, N)\}$$

associated to each minimum, we have

$$\mathcal{Q}_{\Lambda,\beta,\eta}^{\alpha_1,...,\alpha_l}\left( \prod_{i=1}^{k} f_i(m_\Lambda^{\mu_i}(\sigma;\xi))\right)$$
$$= \sum_{Y_\xi \in \Xi_\xi} \left[ \prod_{i=1}^{k} f_i(m_\Lambda^{\mu_i}(\sigma;\xi))\right] \nu_{\Lambda,\beta,\eta}^{\alpha_1,...,\alpha_l}(Y_\xi)$$
$$= \sum_{r=1}^{l} \sum_{Y_\xi \in \Xi_\xi} \left[ \prod_{i=1}^{k} f_i(m_\Lambda^{\mu_i}(\sigma;\xi))\right] \chi_{[Y_\xi \in A^{(\alpha_r)}]} \nu_{\Lambda,\beta,\eta}^{\alpha_1,...,\alpha_l}(Y_\xi)$$
$$+ \sum_{Y_\xi \in \Xi_\xi} \left[ \prod_{i=1}^{k} f_i(m_\Lambda^{\mu_i}(\sigma;\xi))\right] \chi_{[Y_\xi \in (\bigcup_{r=1}^{l} A^{(\alpha_r)})^c]} \nu_{\Lambda,\beta,\eta}^{\alpha_1,...,\alpha_l}(Y_\xi)$$
$$= \sum_{r=1}^{l} t_1^r + t_2 \tag{5.50}$$

We now choose $\omega \in \tilde{\Omega}$. Then by extending the formalism introduced previously in the case $l = 1$ to the case $l > 1$, it is easy to show that

$$\sum_{Y_\xi \in \Xi_\xi} \chi_{[Y_\xi \in (\bigcup_{r=1}^{l} A^{(\alpha_r)})^c]} \nu_{\Lambda,\beta,\eta}^{\alpha_1,...,\alpha_l}(Y_\xi) < \varrho(\eta, N) \tag{5.51}$$

where $\varrho(\eta, N) \downarrow 0$ as $N \uparrow \infty$, since the sum in (5.51) is carried out over a subset where $X_\xi$ is far from each minimum $X_\xi^{*,(\alpha_j)}$. Therefore the term $t_2$ in (5.50) is bounded by

$$|t_2| < c^k \varrho(\eta, N) \tag{5.52}$$

On the other hand, we can rewrite the terms $t_1^r$ as

$$t_1^r = \sum_{Y_\xi \in \Xi_\xi} \left[ \prod_{i=1}^{k} f_i(m_A^{\mu_i}(\sigma; \xi)) - \prod_{i=1}^{k} f_i(\tilde{m}_{A,\beta,\eta}^{\mu_i,\alpha_r}(\xi)) \right] \chi_{[Y_\xi \in A^{(\alpha_r)}]} \nu_{A,\beta,\eta}^{\alpha_1,\ldots,\alpha_l}(Y_\xi)$$
$$+ \left[ \prod_{i=1}^{k} f_i(\tilde{m}_{A,\beta,\eta}^{\mu_i,\alpha_r}(\xi)) \right] \sum_{Y_\xi \in \Xi_\xi} \chi_{[Y_\xi \in A^{(\alpha_r)}]} \nu_{A,\beta,\eta}^{\alpha_1,\ldots,\alpha_l}(Y_\xi) \tag{5.53}$$

where the first term converges to zero as $N \uparrow \infty$ (see the proof of Theorem 3). To deal with the last term in (5.53), notice that by symmetry

$$\lim_{N \uparrow \infty} \sum_{Y_\xi \in \Xi_\xi} \chi_{[Y_\xi \in A^{(\alpha_r)}]} \nu_{A,\beta,\eta}^{\alpha_1,\ldots,\alpha_l}(Y_\xi) = \frac{1}{l}$$

for all $r = 1,\ldots, l$. Thus, the theorem is proven. ∎

## APPENDIX. UNIFORM INTEGRABILITY OF THE FREE ENERGY

As we have shown in Section 4 (see proof of Theorem 1), the convergence of the free energy in the spaces $L^r(\Omega, \Sigma, \mathbb{P})$, $1 \leqslant r < \infty$, directly follows from the uniform integrability of the sequence $\{|F_A(\xi)|^r, N \geqslant 1\}$ of random variables, that is, the following result holds.

**Proposition.** We have

$$\lim_{a \to \infty} \sup_{N \geqslant N_0} \int_{[|F_A(\xi)|^r > a]} |F_A(\xi)|^r \, d\mathbb{P}(\xi) = 0$$

*Proof.* Decomposing the integral

$$I = \int_{[|F_A(\xi)|^r > a]} |F_A(\xi)|^r \, d\mathbb{P}(\xi) \tag{A1}$$

as a sum of integrals over the sets $[ka < |F_A(\xi)|^r \leqslant (k+1)\,a]$, $k \geqslant 1$, we have

$$I \leqslant \sum_{k=1}^{\infty} a(k+1)\, \mathbb{P}(|F_A(\xi)|^r \geqslant ka) \tag{A2}$$

To bound the probabilities in (A2), notice that

$$|F_A(\xi)| \leqslant \max_{\sigma \in \mathcal{S}^A} \beta \frac{|H_A(\sigma; \xi)|}{N} \tag{A3}$$

Therefore

$$\mathbb{P}(|F_\Lambda(\xi)|^r \geqslant ka) \leqslant \mathbb{P}\left(\max_{\sigma \in \mathscr{S}^\Lambda} \frac{|H_\Lambda(\sigma;\xi)|}{N} \geqslant \frac{1}{\beta}(ka)^{1/r}\right)$$
$$\leqslant \sum_{\sigma \in \mathscr{S}^\Lambda} \mathbb{P}\left(\frac{|H_\Lambda(\sigma;\xi)|}{N} \geqslant \frac{1}{\beta}(ka)^{1/r}\right) \tag{A4}$$

and using the exponential Markov inequality,[5]

$$\mathbb{P}(|F_\Lambda(\xi)|^r \geqslant ka) \leqslant \sum_{\sigma \in \mathscr{S}^\Lambda} [\inf_{\eta>0} e^{-\eta N t} \mathbb{E}_\xi(e^{\eta |H_\Lambda(\sigma;\xi)|})] \tag{A5}$$

where we have put $t = (1/\beta)(ka)^{1/r}$ and the expectation is taken with respect to the family $(\xi_i^\mu)_{i=1,\dots,N;\mu=1,\dots,p}$. To estimate the right-hand side of (A5), we rewrite the expectation as

$$\mathbb{E}_\xi(\exp\{\eta\, |H_\Lambda(\sigma;\xi)|\})$$
$$= \mathbb{E}_{\{W^\mu,\mu=1,\dots,p\}}\left(\prod_{\mu=1}^p \mathbb{E}_\xi\left(\exp\left\{\frac{q}{q-1}\left(\frac{\eta}{N}\right)^{1/2} W^\mu \sum_{i=1}^N \left[\delta(\xi_i^\mu) - \frac{1}{q}\right]\right\}\right)\right) \tag{A6}$$

where $W^\mu$, $\mu = 1,\dots,p$, are i.i.d. random variables on $\mathbb{R}$ with standard normal distribution $\mathscr{N}(0, 1)$. We use the following result.

**Lemma.**[8] Let $S_N = \sum_{i=1}^N X_i$, where $X_i = 0$ with probability $p$ and $X_i = 1$ with probability $1-p$. Then for any real number $\lambda$,

$$\mathbb{E}(e^{\lambda(S_N - \mathbb{E}(S_N))}) \leqslant e^{\lambda^2 N/2}$$

Using this lemma, we get

$$\mathbb{E}_\xi(\exp\{\eta\, |H_\Lambda(\sigma;\xi)|\}) \leqslant \mathbb{E}_{\{W^\mu,\mu=1,\dots,p\}}\left(\prod_{\mu=1}^p \exp\left[\eta\left(\frac{q}{q-1}\right)^2 W\mu^2\right]\right)$$
$$= \frac{1}{[1-\eta(q/(q-1))^2]^{p/2}}$$
$$\leqslant \left[\frac{1}{(1-4\eta)^{1/2}}\right]^{N/2} \tag{A7}$$

where we have use that $p < N$ and $q/(q-1) < 2$. Therefore

$$\inf_{\eta>0} e^{-\eta N t} \mathbb{E}_\xi(e^{\eta |H_\Lambda(\sigma;\xi)|}) \leqslant \inf_{\eta>0} e^{-N f(\eta, t)} \tag{A8}$$

where $f(\eta, t) = \eta t + \frac{1}{2}\ln(1-4\eta)$ is a convex function of $\eta$ maximized at the point $\eta^* = -1/2t + 1/4$. Remembering that $t = (1/\beta)(ka)^{1/r}$, we finally have that for all $a > a_0$ and $a_0$ sufficiently large,

$$\inf_{\eta > 0} e^{-\eta N t}\mathbb{E}_\xi(e^{\eta |H_A(\sigma;\xi)|}) \leqslant e^{-N(c/4\beta)(ka)^{1/r}} \tag{A9}$$

where $c$ is a positive constant depending on $a_0$ and $\beta$. Putting (A9) together with (A5), we find

$$\mathbb{P}(|F_A(\xi)|^r \geqslant ka) \leqslant q^N e^{-N(c/4\beta)(ka)^{1/r}} \tag{A10}$$

The proposition is now proven by inserting (A10) in (A2) and taking successively the supremum over $N > N_0$ and the limit $a \uparrow \infty$. ∎

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